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Mathematics Upgrading

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Mathematics Upgrading

Lessons A-K

Distance Learning

Alberta
EDUCATION

Mathematics Upgrading
Student Module
Lessons A-K and Answers
Alberta Distance Learning Centre
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IMPORTANT

PLEASE READ THIS INTRODUCTION BEFORE BEGINNING LESSON A.

Content and Purpose of Course

Mathematics Upgrading consists of eleven lessons. These lessons cover important background material from Mathematics 10 and 20 that the students must be familiar with before they can handle Mathematics 30 or Mathematics 33.

Mathematics Upgrading is specifically designed for those students who have not completed Mathematics 20, students who have poor gradings in Mathematics 20, and students who have completed Mathematics 20 but have been out of school for a few years.

The following list provides a brief outline of the topics covered in Lessons A-K.

Lesson	Topics
A	Set Theory, Properties of Whole Numbers
B	The Set of Real Numbers and Its Subsets
C	Powers, Roots, Variable Expressions
D	Polynomials
E	Factoring: Rational Expressions
F	Equations and Inequalities In One Variable
G	Linear Equations in Two Variables, Relations, Functions, Linear Functions
H	Quadratic Functions and Relations
I	Complex Numbers, Equation Systems
J	Geometry
K	The Metric System, Trigonometry, Statistics

Textbooks

There is no prescribed textbook that accompanies the lessons of Mathematics 30A. All the explanatory material and exercises are provided in the lessons.

Self-correcting Exercises

A number of self-correcting exercises appear in the lessons of Mathematics Upgrading. Each of these exercises should be worked through carefully and then checked with the answers provided at the end of the lesson. If you do not understand some of the answers to these exercises, ask your Correspondence teacher about them. Write your questions beside the problems you do not understand.

General Instructions

1. Remove the staples from your set of lessons and transfer the lesson pages to a looseleaf binder which will preserve both the unused and corrected lessons. Do not tear the pages from the book.
2. Do not skip any reading material. Several readings may be needed in mathematics before you can expect to grasp some of the ideas. Make sure that you fill in any blanks that appear in the reading material. These blanks are intended to keep you alert and check your understanding of previous reading.
3. Do the exercises and lessons in the proper order.
4. We recommend that you use ink for all your work except diagrams and graphs.
5. In order to complete Lessons J and K in this course you will need a ruler and protractor. Obtain these items before you begin these lessons.

This upgrading course is a self-study course in which you will correct your own answers. These answers are given in the Answer Key. Appropriate explanations are also supplied. **Please do not submit these lessons to the Alberta Distance Learning School for correction.**

In order for effective learning to occur, study the lesson notes very carefully so that the concepts are understood. Once the material is fully comprehended, complete the questions, and then check the solutions and answers in the booklet provided. Any questions which you found difficult or could not do should be carefully noted. Study the answers provided, and also review the appropriate background notes in the lesson.

If additional assistance is needed, you may telephone the Alberta Distance Learning School at 673-5333. Students who reside in Alberta may call the school free of charge through the local RITE number in your area or the Zenith number 22 333, if there is no RITE centre in your calling area.

Lesson A

SET THEORY; PROPERTIES OF WHOLE NUMBERS

Topic One: Basic Set Theory

In everyday life we use many different words to express the idea of a collection or group of objects. Thus, we speak of a herd of cattle, a pack of wolves, a flock of sheep, a swarm of bees, a stamp collection or a set of golf clubs.

In mathematics, we use the term SET to refer to a well-defined collection of objects. By "well-defined", we mean that it is possible to tell without doubt whether or not a particular object belongs to the set.

A. Specifying Sets

Any object that belongs to a given set is called a MEMBER or ELEMENT of that set. There are various ways that have been developed for referring to the elements of a set. Let us look at some of these.

I. Tabulation

The elements of a set may be written inside a set of curly braces. Each member of the set is listed only once. Commas are used to separate the elements and these elements may be listed in any order. When the elements of a set are listed within a pair of braces, we say that the set has been TABULATED. For convenience, we choose some capital letter to represent the set.

EXAMPLE 1: Tabulate the set of numbers less than 18 that are exactly divisible by 5. Let the letter A represent this set.

Solution

The numbers 5, 10, and 15 must belong to this set since they are the only numbers that are less than 18 and are exactly divisible by 5. Thus, set A could be tabulated as follows:

$$\underline{A = \{5, 10, 15\}}$$

This is read, "A is the set whose elements are 5, 10, and 15." Note that this set could also be tabulated as $\{5, 15, 10\}$, $\{10, 5, 15\}$, $\{10, 15, 5\}$, $\{15, 10, 5\}$, or $\{15, 5, 10\}$, since the order of listing the elements is not important.

EXAMPLE 2: Tabulate the set of letters in the word "Mississippi".

Solution

Note that the word "Mississippi" contains only 4 different letters. Remember that the elements of a set should be listed only once. Thus, this set could be tabulated as follows:

$$\underline{L = \{m\ ,\ i,\ s,\ p\}}$$

The symbol "$\in$", which is the Greek letter "epsilon", is used to mean "is an element of". We draw a line through the letter epsilon to arrive at the symbol "$\notin$" which means "is not an element of."

EXAMPLE: $B = \{3, 9, 7, 2\}$

Since 2 is a member of set B, we can write "$2 \in B$".
(This is read, "2 is an element of set B.")
Since 5 is not a member of set B, we can write "$5 \notin B$".
(This is read, "5 is not an element of set B.")

Write in symbols:

3 is an element of set B. ____________

8 is not an element of set B. ____________

Self-correcting Exercise #1

Answers may be found on page 44 of this lesson.

1. Tabulate each of the following sets. (Remember to use curly braces around the sets and place commas between the elements listed.)

   (a) P, the set of prairie provinces in Canada.

   ____

   (b) C, the set of numerals on the face of a clock.

   ____

   (c) V, the set of vowels in the English alphabet. (Do not include "y".)

   ____

   (d) O, the set of odd one-digit numbers.

   ____

2. $A = \{2, 4, 6, 8, 10\}$ and $B = \{5, 6, 7, 8, 9, 10\}$.
Insert either the symbol "∈" or "∉" to make each of the following statements true. (Remember that the symbol "∈" means " is an element of" and the symbol "∉" means "is not an element of.")

(a) 2 _____ A

(b) 2 _____ B

(c) 6 _____ A

(d) 6 _____ B

(e) 9 _____ A

(f) 9 _____ B

When tabulating sets of numbers, it is sometimes difficult or time-consuming to list all the members of the set. In cases in which the elements follow a regular sequence or pattern, we can overcome this obstacle by using three dots "..." to indicate omitted members. You must go through the following steps in order to tabulate a set with omitted members.

(i) List the first three elements so that the pattern is clearly established. (Put a comma after each element listed.)

(ii) Use three dots (followed by a comma) to indicate that the sequence continues in the same fashion.

(iii) Give the last element if it is known.

EXAMPLE 1: Tabulate E, the set of even numbers less than 100.
Solution

The first three elements in this set are 2, 4, and 6, and the last element is 98. We can use three dots to indicate omitted elements.

$$E = \{2, 4, 6, \ldots, 98\}$$

COMMAS APPEAR BEFORE AND AFTER THE 3 DOTS.

EXAMPLE 2: Tabulate M, the set of multiples of 5.
Solution

The first three multiples of 5 are 5, 10, and 15.
This set goes on indefinitely, with no last member.

$$M = \{5, 10, 15, \ldots\}$$

In mathematics, one of the most important number sets that you will deal with is the set of NATURAL NUMBERS 1, 2, 3, 4, 5, and so on. We always represent this set by the capital letter N and tabulate it as follows:

$$N = \{1, 2, 3, \ldots\}$$

THE THREE DOTS MEAN "AND SO ON".

Self-correcting Exercise #2

Answers may be found on page 44 of this lesson.

1. Tabulate each set, using three dots to indicate omitted members.

   (a) A, the set of natural numbers between 3 and 104.
   (Hint: 3 and 104 are not included.)

   ____

   (b) B, the set of odd natural numbers greater than or equal to 33.

   ____

   (c) C, the set of even natural numbers between 12 and 86 inclusive.
   (Hint: The word "inclusive" means that 12 and 86 are included.)

   ____

   (d) D, the set of all two-digit natural numbers.

   ____

2. If $S = \{3, 6, 9, \ldots, 30\}$, decide if each of the following statements is true or false.

   (a) $11 \in S$ false

   (b) $5 \in S$ __________

   (c) $15 \in S$ __________

   (d) $27 \in S$ __________

   (e) $29 \in S$ __________

   (f) $18 \in S$ __________

II. Rule Method

Sometimes we specify sets by stating within a pair of braces the conditions under which objects belong to the set. The rule method is very useful for specifying sets which are tedious to tabulate. For example, if we want to specify the set of cities in North America, it would be impractical to tabulate this set. Also, three dots could not be used to indicate omitted members since the elements of this set do not follow any regular sequence or pattern. This set could be specified by the rule method as follows:

$$C = \{\text{all cities in North America}\}$$

Specify each of the following sets using the rule method.

1. A = {1, 2, 3, ..., 50}

   A = {all natural numbers less than 51}

2. B = {2, 4, 6, ...}

   ____

3. C = {6, 12, 18, ...}

   ____

4. D = {January, June, July}

   ____

III. Set-Builder Notation

When dealing with sets of numbers in mathematics, we often use SET-BUILDER NOTATION which is a refinement of the rule method of specifying sets. This notation has the following features:

1. The set-builder symbol $\{x|\qquad\}$ is used. It is read, "the set of all x such that ...". The word "all" is used in reading this notation even though it is not actually written. The vertical line " | " which is called a SOLIDUS is a handy abbreviation for the words "such that".
2. In the empty space that appears after the solidus, we mention a property or characteristic common to all members of the set.

In stating the common characteristics of the elements of a set, we must often make use of the symbols for "greater than" or "less than". Learn the meanings of the following symbols.

SYMBOL	MEANING
$<$	is less than
$>$	is greater than
$\leqq$	is less than or equal to
$\geqq$	is greater than or equal to

Thus, $x < 5$ means that x is less than 5.
$x \geqq 2$ means that x is greater than or equal to 2.

How would you write:

(i) x is greater than 7? ____________

(ii) x is less than or equal to 17? ___________

Now, suppose we wished to specify the set of all natural numbers greater than or equal to 9 using set-builder notation. We could write $x \geqq 9$ to indicate that all elements of the set must be greater than or equal to 9. We could write $x \in N$ to indicate that all elements of the set must be natural numbers. Thus, this set could be specified in set-builder notation as follows:

$$\{x \mid x \geqq 9,\ x \in N\}$$

This is read, "the set of all x such that x is greater than or equal to 9 and x is a natural number". This set contains the natural numbers 9, 10, 11, and so on.

Use set-builder notation to specify the set of all natural numbers less than 15.

{___x|________________,___________}

Now tabulate this set. (Use 3 dots to indicate omitted members.)

{_________________________________}

Sometimes we are told that the elements of a set of numbers lie between two boundaries. Then we must use arrowheads that point to the left on either side of x. We place the smaller boundary to the left of x and the upper boundary to the right of x. Note how the following statements are written symbolically.

Statement	Symbolic Representation
1. x is between 3 and 11.	smaller number → $3 < x < 11$ ← larger number Both arrowheads point to the left. This can be read, "x is greater than 3 and less than 11."
2. x is between 2 and 7 inclusive.	$2 \leqq x \leqq 7$ This can be read, "x is greater than or equal to 2 and less than or equal to 7."

3. x is greater than or equal to 6 and less than 12.	$6 \leqq x < 12$
4. x is greater than 4 and less than or equal to 9.	$4 < x \leqq 9$

How would you write:

(i) x is greater than 1 and less than 9? ____________

(ii) x is less than or equal to 50 and greater than 6? ____________

(iii) x is between 4 and 40 inclusive? ____________

(iv) x is between 3 and 25? ____________

Now, let us specify the following sets using set-building notation.

1. The set of all natural numbers between 3 and 25.

   Set-builder notation: $\{x \mid 3 < x < 25,\ x \in N\}$
   This is read, "the set of all x such that x is greater than 3 and less than 25 and x is a natural number."

2. The set of all natural numbers between 5 and 10 inclusive.

   Set-builder notation: $\{x \mid$ ____________, $x \in$ ___$\}$.
   This is read, "the set of all x such that x is greater than or equal to 5 and ____________________, and x is a natural number."

3. The set of all natural numbers greater than or equal to 2 and less than 7.

   Set-builder notation: ____________________

   This is read, " ____________________

   ____________________________________"

Self-correcting Exericse #3

Answers may be found on page 44 of this lesson.

1. Write the following statements in symbols.

   (a) x is less than 35. $x < 35$

   (b) x is between 12 and 48. ______

   (c) x is greater than or equal to 5. ______

   (d) x is greater than 7 and less than or equal to 54. ______

   (e) x is between 4 and 16 inclusive. ______

   (f) x is less than or equal to 6. ______

2. Specify each set using set-builder notation. Then tabulate the set.

   (a) The set of all natural numbers greater than 6.

   Set-builder notation: ______

   Tabulation: ______

   (b) The set of all natural numbers greater than or equal to 3 and less than 33.

   Set-builder notation: ______

   Tabulation: ______

   (c) The set of all natural numbers less than or equal to 30.

   Set-builder notation: ______

   Tabulation: ______

   (d) The set of all natural numbers between 4 and 82.

   Set-builder notation: ______

   Tabulation: ______

B. Finite, Infinite, and Null Sets

I. Finite Sets

If it is possible to list or count all the elements of a set, the set is called a FINITE SET. For example, the set of provinces in Canada is a finite set since it has ten members which can be counted. The set $\{1, 2, 3, \ldots, 100\}$ is also a finite set since it has a limited number of elements (100 to be exact).

Tabulate another finite set. ________________________________

How many elements does this set have? ________

II. Infinite Sets

If a set has an unlimited number of elements it is called an INFINITE SET. For example, the set of even natural numbers $\{2, 4, 6, \ldots\}$ is an infinite set. The three dots indicate that the sequence continues but there is no last element. When dealing with infinite sets, regardless of the number of members you list, there will always be another member than can be listed.

Tabulate another infinite set. ________________________________

III. The Null Set (Empty Set)

The set of all natural numbers between 4 and 5 contains no members. Similarly, there are no elements in the set of all women who have been Prime Minister of Canada. A set which contains no members is called an EMPTY SET and is referred to as the NULL SET. The null set is represented by a pair of empty braces, $\{\ \}$, or by the Greek letter phi, ϕ .

Thus,

$\{x \mid 4 < x < 5, x \in N\} = \{\ \}$ or ϕ.
$\{$all women who have been Prime Minister of Canada$\} = \{\ \}$ or ϕ .

Specify another empty set. ________________________________

Self-correcting Exercise #4

Answers may be found on page 45 of this lesson.

1. Classify each of the following sets as being finite, infinite, or null.

   (a) $\{1, 3, 5, \ldots, 135\}$ ____________

   (b) $\{1, 3, 5, \ldots\}$ ____________

   (c) $\{0\}$ ____________

   (d) $\{\ \}$ ____________

   (e) {all rivers in the world} ____________

   (f) $\{x \mid x < 12, x \in N\}$ ____________

   (g) {all months of the year having 32 days} ____________

   (h) $\{x \mid x \geqq 5, x \in N\}$ ____________

C. Subsets

If every element of set A is an element of Set B, then set A is a SUBSET of set B; that is, set A is included in set B.

NOTATION:

$A \subseteq B$	This is read "set A is a subset of set B."
$A \nsubseteq B$	This is read "set A is not a subset of set B." (This means that set A contains at least one element which is not an element of set B.)

EXAMPLES:

1. The set of lakes in Alberta is a subset of the set of lakes in Canada.

2. The set of names beginning with S in the Edmonton telephone directory is a subset of the set of all the names in the Edmonton directory.

3. If $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{2, 3, 4\}$; we could say that set A is a subset of set B and write $A \subseteq B$. Note that the numbers 1, 2, and 3 that are elements of set A are all elements of set B. Note that set A is not a subset of set C ($A \nsubseteq C$) since the number 1 which belongs to set A is not an element of set C.

Is set B a subset of set C? ______ Why? ____________________

____

Is set C a subset of set B? ______ Why? ____________________

____

Is set C a subset of set A? ______ Why? ____________________

____

4. If $R = \{a, b, c\}$ we could say that ϕ is a subset of set R. In general, the null set is a subset of every set. Also, $\{a, b, c\}$ is a subset of set R. In general, every set is a subset of itself.

Suppose we endeavored to form all the possible subsets of the set $A = \{1, 2, 3\}$. What would these subsets be and how many would there be?

First, in forming subsets of A, we could choose any one of the three members of A. This would give us the three subsets $\{1\}$, $\{2\}$, and $\{3\}$.

Next, we could choose any two of the three members of A. This would give us the three subsets $\{1, 3\}$, $\{1, 2\}$ and $\{2, 3\}$. (Remember - the order of listing the elements in a set is not important.)

We could also choose all three of the three members of A and obtain the subset $\{1, 2, 3\}$.

But we must also consider the case in which we choose none of the members of A. This would give us the empty set, $\{ \}$.

Thus, we can form eight possible subsets of the set $A = \{1, 2, 3\}$. These eight subsets are:

$\{1\}$, $\{2\}$, $\{3\}$, $\{1, 3\}$, $\{1, 2\}$, $\{2, 3\}$, $\{1, 2, 3\}$, $\{ \}$.

In general, how many subsets has a set containing n elements? Let us try to answer this question by forming subsets of sets containing 0, 1, 2, 3, or 4 elements.

Fill in the blanks in the following table.

Set	No. of Elements in Set	Subsets	Number of Subsets
{ }	0	{ }	1
{a}	1	{ }, {a}	________
{a, b}	________	{ }, {a}, ____, ____	________
{a, b, c}	________	____________ ____________	________
{a, b, c, d}	________	{ }, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c} ________ ____________ ____________	16

Thus, a set with 0 elements has $2^0 = 1$ subset
a set with 1 element has $2^1 = 2$ subsets

a set with 2 elements has $2^2 =$ ______ subsets

a set with 3 elements has ____________

a set with 4 elements has ____________

Conclusion: A set with n elements has 2^n subsets.

Note: In the expression 2^n, the exponent n tells you how many times 2 is to be used as a factor. For example,

$2^3 = 2 \times 2 \times 2 = 8$ ← 2 is used as a factor 3 times.
$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ ← 2 is used as a factor 5 times.

By definition, $2^0 = 1$.

Self-correcting Exercise #5

Answers may be found on page 45 of this lesson.

1. $R = \{2, 4, 6, \ldots, 32\}$
Look at each set below and decide whether or not it is a subset of set R. Write "yes" in the blank if it is and "no" if it isn't.

(a) $\{2, 4, 6\}$ ________
(b) $\{7, 8, 9, 10\}$ ________
(c) $\{2\}$ ________

(d) $\emptyset$ ________
(e) $\{6, 12, 16, 28\}$ ________
(f) $\{20, 21, 22, \ldots, 32\}$ ________

(g) $\{2, 4, 6, \ldots, 32\}$ ________
(h) $\{0\}$ ________
(i) $\{16, 18, 20, \ldots, 30\}$ ________

2. How many subsets can be formed from each of the following sets?

Set	Number of subsets
(a) $\{x \mid x < 6, x \in N\}$	$2^5 = 32$
(b) $\{1, 3, 5, 7\}$	________
(c) $\{1\}$	________
(d) {red, orange}	________
(e) {Paul, Bill, Jack}	________

3. List all the possible subsets of {blue, green, red}.

____

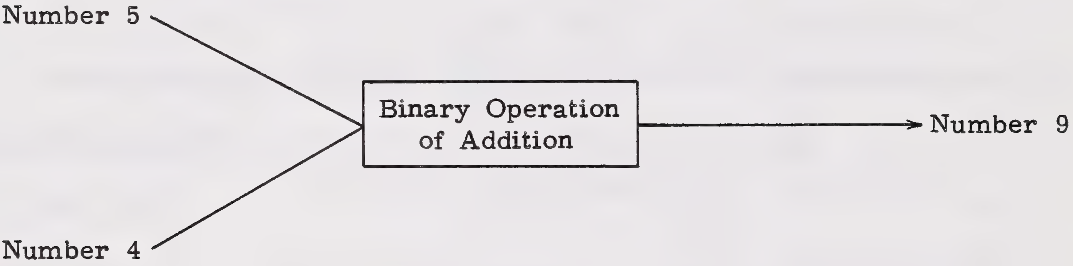
____

D. Operations on Sets

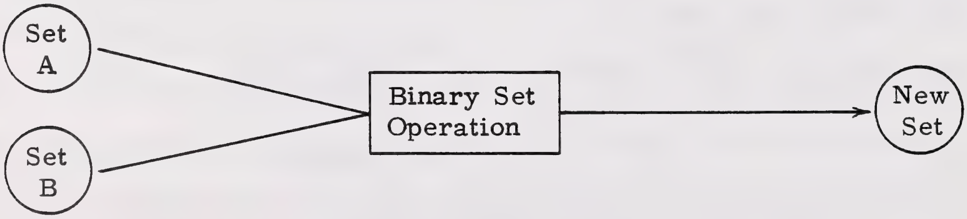
In your mathematics experience up to this point, you have always performed operations on individual members of a set. For example, in elementary arithmetic you learned to perform the operations of addition, subtraction, multiplication, and division on members of number sets.

Now we come to a new type of operation which is performed on one or more entire sets rather than on particular members of a set. This type of operation is called a SET OPERATION.

In arithmetic, the familiar operations of addition, subtraction, multiplication and division are called BINARY OPERATIONS because they are operations that are performed on two numbers at a time in order to produce a third number. For example, we can combine the two numbers 5 and 4 by the binary operation of addition in order to arrive at a third number 9.



Set operations which are performed on two sets of objects are called binary operations on sets. In all binary set operations, we begin with two sets, perform the given operation, and produce a third set.



In the following sections, we will discuss three binary set operations; INTERSECTION, UNION, and CARTESIAN PRODUCT.

I. Intersection

DEFINITION: The INTERSECTION of set A and set B is the set of all elements that are common to both set A and set B.

NOTATION: $A \cap B \leftarrow$ This is read, "A intersection B".

EXAMPLE: If $A = \{1, 3, 5, 7, 9\}$ and $B = \{3, 4, 5, 6\}$ find $A \cap B$.

Solution

What elements do set A and set B have in common? Since the elements 3 and 5 belong to both sets, they must belong to the intersection set, $A \cap B$.

i.e. $A \cap B = \{3, 5\}$

THESE ELEMENTS BELONG TO BOTH SETS.

If two sets have no elements in common, their intersection is { }, the null set. Two such sets are said to be DISJOINT.

EXAMPLE: If $R = \{2, 4, 6\}$ and $S = \{8, 10\}$, find $R \cap S$.

Solution

Since R and S have no elements in common, they are disjoint sets.

$$R \cap S = \{\ \}$$

NOTE: An intersection set can also be found for more than two sets. The intersection set $A \cap B \cap C$ will contain all the elements common to all three sets. The intersection set $A \cap B \cap C \cap D$ will contain all the elements common to all four sets ... and so on.

EXAMPLE: If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 6, 9, 10\}$, and $C = \{6, 12, 24, 48\}$, Find $A \cap B \cap C$.

Solution

$$\begin{aligned} A \cap B \cap C &= \{1, 2, 3, 4, 5, 6\} \cap \{3, 6, 9, 10\} \cap \{6, 12, 24, 48\}, \\ &= \{3, 6\} \cap \{6, 12, 24, 48\} \\ &= \{6\} \end{aligned}$$

The three sets A, B, and C have the one element 6 in common.

Self-correcting Exercise #6

Answers may be found on page 46 of this lesson.

1. For each exercise, decide what elements the sets have in common. Then tabulate the intersection set. (Remember to put curly braces around each intersection set.)

   (a) $A = \{6, 7, 8, 9\}$ and $B = \{7, 9, 13\}$
   What elements do A and B have in common? ________

   $A \cap B =$ ____________________

   (b) $R = \{12, 14, 16, 18\}$ and $S = \{16, 14, 18, 12\}$
   What elements do R and S have in common? ________

   $R \cap S =$ ____________________

   (c) $M = \{4, 5, 6\}$ and $N = \{9, 10\}$
   What elements do M and N have in common? ________

   $M \cap N =$ ____________________

   (d) $J = \{2, 4, 6, \ldots, 20\}$ and $K = \{4, 8, 12, 24, 26\}$
   What elements do J and K have in common? ________

   $J \cap K =$ ____________________

   (e) $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$, and $C = \{2, 3, 4\}$
   What elements do A, B, and C have in common? ________

   $A \cap B \cap C =$ ____________________

II. Union

DEFINITION: The UNION of set A and set B is the set of all elements that belong to set A, set B, or both.

NOTATION: $A \cup B \longleftarrow$ This is read, "A union B".

EXAMPLE: If $A = \{1, 4, 6, 9, 10\}$ and $B = \{4, 5, 6, 9, 10, 12, 15\}$, find $A \cup B$.

Solution

What elements belong to set A, or set B, or both? Note that the elements 1, 4, 6, 9, and 10 belong to set A so must also belong to the union set. Now, look at the elements in set B. Since the elements 4, 6, 9, 10 have already been placed in the union set, we need only add the elements 5, 12, and 15.

i.e. $A \cup B = \{1, 4, 5, 6, 9, 10, 12, 15\}$

NOTE: A union set can also be found for more than two sets. The union set $A \cup B \cup C$ will contain all the elements that belong to any of the three sets. The union set $A \cup B \cup C \cup D$ will contain all the elements that belong to any of the four sets ... and so on.

EXAMPLE: If $A = \{1,2,3,4,5,6\}$, $B = \{2,4,6,8,10\}$, and $C = \{1,3,5,7,9\}$, find $A \cup B \cup C$.

Solution

$$\begin{aligned} A \cup B \cup C &= \{1,2,3,4,5,6\} \cup \{2,4,6,8,10\} \cup \{1,3,5,7,9\} \\ &= \{1,2,3,4,5,6,8,10\} \cup \{1,3,5,7,9\} \\ &= \{1,2,3,4,5,6,7,8,9,10\} \end{aligned}$$

Self-correcting Exercise #7

Answers may be found on page 46 of this lesson.

1. For each exercise, state the elements that set A contributes to the union set, state the new elements that set B contributes, and then tabulate $A \cup B$. (Remember to put curly braces around each union set.)

(a) $A = \{6,7,8,9\}$ and $B = \{7,9,13\}$

Elements which set A contributes ______

New elements which set B contributes ______

$A \cup B =$ ______

(b) $M = \{4,5,6\}$ and $N = \{9,10\}$

Elements which set M contributes ______

New elements which set N contributes ______

$M \cup N =$ ______

(c) $R = \{4,5,6\}$ and $S = \{1,3,4,5,6,8,9\}$

Elements which set R contributes ______

New elements which set S contributes ______

$R \cup S =$ ______

(d) $P = \{1,2,3,\ldots,15\}$ and $Q = \{8,9,10,\ldots,18\}$

Elements which set P contributes ______

New elements which set Q contributes ______

$P \cup Q =$ ______

2. If $A = \{b, c, d\}$, $B = \{a, c, f, g\}$ and $C = \{a, c, f, i, l\}$, find $A \cup B \cup C$.

Elements which set A contributes ____________

New elements which set B contributes ____________

New elements which set C contributes ____________

$A \cup B \cup C =$ ________________________________

IMPORTANT: At first you may have difficulty distinguishing between the symbols for intersection and union. Here is one way to remember them.

Intersection: The key word in describing an intersection is "AND." You are interested in elements that are common to both set A AND set B. The symbol for intersection, $\cap$, is like an A with the crossbar missing.

Union: The symbol for union, $\cup$, is similar to the first letter in the word "Union." Historically, the symbol $\cup$ evolved from the letter v; v is the first letter of the Latin word "vel" which means "OR." "OR" is the key word in describing a union since you are concerned with the elements that belong to set A OR set B.

III. Cartesian Product

DEFINITION: The CARTESIAN PRODUCT of set A and set B is the set of ordered pairs that can be formed by matching each member of set A in turn with each member of set B. The members of set A are the first components of the ordered pairs and the members of B are the second components.

NOTATION: $A \times B$ ← This is read, "A cross B".

An ORDERED PAIR is two objects which occur in a special order. The object which appears first is called the first component of the ordered pair. The object which appears second is called the second component. Components of ordered pairs are always written inside a set of parentheses. A comma is placed between the two components. For example, (x, y) is an ordered pair. It is referred to as the ordered pair x, y. The first component of this ordered pair is x and the second component is y. Note that (y, x) is an entirely different ordered pair. In this case, the first component is y and the second component is x.

EXAMPLE: An experiment consists of tossing a coin and rolling a "die" (singular of "dice") and noting the pair of outcomes that is obtained. In tossing a coin, you can either obtain a "head" (H) or a "tail" (T). We can consider these outcomes as being members of set A.

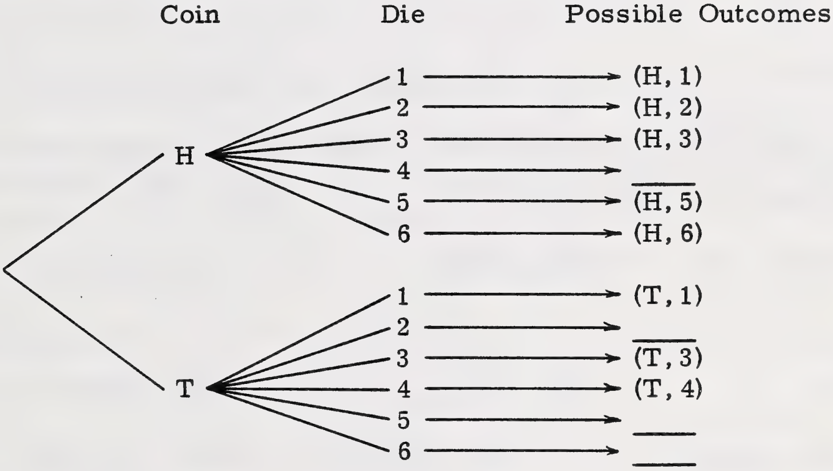
$$A = \{H, T\}$$

In rolling a die, you can obtain any number from 1 to 6 inclusive. We can consider these outcomes as being members of set B.

$$B = \{1, 2, 3, 4, 5, 6\}$$

We can form the Cartesian product A × B in order to determine the possible pairs of outcomes. A TREE DIAGRAM can be used to find all the ordered pairs that belong to A × B. In the tree diagram below, note that each member of set A is paired in turn with each member of set B.

Fill in the blanks in the tree diagram below.



The Cartesian set A × B could be tabulated as follows:

$$A \times B = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),\\ (T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$

Each member of set A × B is an ordered pair whose first component tells you the outcome of tossing the coin and second component tells you the outcome of rolling the die.

For example, the ordered pair (H, 4) tells you that a head was obtained on the coin and a 4 rolled on the die.

What does the ordered pair (T, 2) tell you?

____

Which ordered pairs correspond to the outcome of obtaining a head and a number greater than 3?

____

Self-correcting Exercise #8

Answers may be found on page 46 of this lesson.

1. If $A = \{1, 3, 5\}$ and $B = \{5, 6\}$, tabulate each of the following sets. (Remember to put round brackets around each ordered pair and braces around the entire tabulation.)

   (a) $A \times B =$ ______________________________

   (b) $B \times A =$ ______________________________

   (c) $A \times A =$ ______________________________

   (d) $B \times B =$ ______________________________

2. If $R = \{1, 3, 5, 7\}$ and $S = \{3, 4, 5\}$, decide whether or not each of the following ordered pairs belongs to $R \times S$. Write "yes" in the blank if it does and "no" if it doesn't.

   (a) (1, 3) __________ (b) (3, 1) __________ (c) (3, 2) __________

   (d) (3, 3) __________ (e) (1, 1) __________ (f) (7, 4) __________

   (g) (5, 3) __________ (h) (5, 7) __________ (i) (2, 5) __________

EXERCISE - Basic Set Theory

1. Fill in the blanks.

(a) The symbol $\in$ means ________________________.

(b) When we list the elements of a set within a pair of braces, we call this ________________________ the set.

(c) $N = \{1, 2, 3, \ldots\}$ is the set of ____________________ numbers.

(d) When the set of all natural numbers less than 3 is written as $\{x \mid x < 3,\ x \in N\}$, it is said to be in ______________________ notation. In this notation, the symbol $\mid$ is read "______________."

(e) A set with a limited number of elements is called a ______________________ set.

(f) A set with no members is called the ______________ set or the ____________ set. It can be represented either by the symbol ______ or the symbol ______.

(g) If every element of set R also belongs to set S, then set R is a ____________________ of set S.

(h) $R \cup S$ is read, "R ______________ S".

(i) A Cartesian product is a set of ____________________ pairs.

(j) Three binary operations on sets are ______________________, ____________________, and ______________________.

2. Tabulate each of the following sets. (Where necessary, use 3 dots to indicate omitted members.) Then, tell whether each set is finite, infinite, or null.

(a) $A = \{x \mid 7 \leqq x < 53,\ x \in N\}$

Tabulation: $A = \{7, 8, 9, \ldots, 52\}$

Classification: finite

(b) B, the set of all the months of the year that have neither 30 nor 31 days.

Tabulation: ______________________________

Classification: ______________________________

(c) C = {all natural numbers that are exactly divisible by 7.}

Tabulation: ______________________________

Classification: ______________________________

(d) $D = \{x \mid 1 < x < 2,\ x \in N\}$

Tabulation: ______________________________

Classification: ______________________________

(e) E, the set of days in the week.

Tabulation: ______________________________

Classification: ______________________________

(f) $F = \{x \mid x > 4,\ x \in N\}$

Tabulation: ______________________________

Classification: ______________________________

(g) G, the set of letters in the word "missiles".

Tabulation: ______________________________

Classification: ______________________________

(h) H = {all the months of the year beginning with the letter "C"}

Tabulation: ______________________________

Classification: ______________________________

(i) $I = \{x \mid 5 \leqq x \leqq 135,\ x \text{ is a multiple of } 5\}$

Tabulation: ______________________________

Classification: ______________________________

(j) J, the set of all 2-digit numbers that are squares of 1-digit numbers.

Tabulation: ______________________________

Classification: ______________________________

(k) K, the set of odd natural numbers.

Tabulation: ______________________________

Classification: ______________________________

3. Describe the error in each of the following tabulations.

(a) The set of letters in the word "better" is L = {b, e, t, t, e, r}.

____

____

(b) The set of odd natural numbers less than 9 is A = {1, 3, 5, 7, 9}.

____

____

(c) The set of letters in the word "blue" is B = {b l u e}.

____

____

(d) The set of two-digit counting numbers ending in zero is C = {10, 20, 30, 40, 50, 60, 70, 80, 90, 100}.

____

____

4. Specify each of the following sets using set-builder notation. Then tabulate the set.

(a) The set of all natural numbers between 4 and 24 inclusive.

Set-builder notation: $\{x \mid 4 \leqq x \leqq 24, x \in N\}$

Tabulation: $\{4, 5, 6, \ldots, 24\}$

(b) The set of all natural numbers greater than 12.

Set-builder notation: ____________________

Tabulation: ____________________

(c) The set of all natural numbers greater than 9 and less than or equal to 36.

Set-builder notation: ____________________

Tabulation: ____________________

(d) The set of all natural numbers between 5 and 8.

Set-builder notation: ____________________

Tabulation: ____________________

(e) The set of all natural numbers less than or equal to 100.

Set-builder notation: ____________________

Tabulation: ____________________

(f) The set of all natural numbers greater than or equal to 3 and less than 42.

Set-builder notation: ____________________

Tabulation: ____________________

5. Tabulate the following union and intersection sets.

(a) $A = \{1, 2, 9\}$

$B = \{2, 9, 13, 17, 20\}$

$A \cup B =$ $\{1, 2, 9, 13, 17, 20\}$

$A \cap B =$ $\{2, 9\}$

(b) $A = \{1, 2, 3, 4\}$

$B = \{1, 4, 9, 25, 36\}$

$A \cup B =$ ____________________

$A \cap B =$ ____________________

(c) $A = \{\text{black}, \text{green}\}$

$B = \{\text{red}, \text{yellow}\}$

$A \cup B =$ ____________________

$A \cap B =$ ____________________

(d) $A = \{3, 4, 5, 6\}$

$B = \{4, 5\}$

$A \cup B =$ ____________________

$A \cap B =$ ____________________

(e) $A = \{1, 2, 3\}$

$B = \{\ \ \}$

$A \cup B =$ ____________________

$A \cap B =$ ____________________

(f) $A = \{15, 20, 25, 27, 51, 60\}$

$B = \{15, 26, 60, 75, 100\}$

$A \cup B =$ ____________________

$A \cap B =$ ____________________

MUST BE INFINITE SINCE BOTH A AND B ARE INFINITE SETS.

(g) $A = \{1, 3, 5, \ldots\}$

$B = \{2, 4, 6, \ldots\}$

$A \cup B =$ ____________________

$A \cap B =$ ____________________

INFINITE SETS

(h) $A = \{1, 2, 3, \ldots\}$

$B = \{2, 4, 6, \ldots\}$

$A \cup B =$ ____________________

$A \cap B =$ $\{2, 4, 6, \ldots\}$

(i) $A = \{5, 3, 2\}$

$B = \{2, 3, 5\}$

$A \cup B =$ ____________________

$A \cap B =$ ____________________

(j) $A = \{3, 4, 5\}$ $A \cup B \cup C =$ ________

$B = \{1, 2, 3\}$ $A \cap B \cap C =$ ________

$C = \{3, 5, 7\}$

(k) $A = \{a, b, c, d\}$ $A \cup B \cup C =$ ________

$B = \{c, d\}$ $A \cap B \cap C =$ ________

$C = \{b, c, d, e\}$

6. List all the possible subsets of $A = \{7, 8, 9\}$.

____

7. $S = \{10, 20, 30, 40, 50, 60\}$

(a) List four different subsets of S that contain exactly 4 elements.

{20, 40, 50, 60}, ________, ________, ________

(b) List three different subsets of S that contain exactly 5 elements.

________, ________, ________

(c) List all the possible subsets of S that contain exactly 1 element.

____

(d) How many subsets can be formed from set S? ________

8. If $A = \{3, 4\}$, $B = \{2, 4, 6\}$ and $C = \{1, 2, 3, 4, 5\}$, tabulate the following sets.

(a) $A \times B = \{$ ________

(b) $B \times A = \{$ ________

(c) $B \times B = \{$ ________

(d) $A \times C = \{$ ________

____

(e) $A \cap B \cap C =$ ________

(f) $A \cup B \cup C =$ ________

9. $A = \{2, 4, 6, 8, 10\}$, $B = \{4, 6, 8, 10\}$, $C = \{2, 3, 4\}$ and $D = \{4, 8\}$.
Decide whether each of the following statements is true or false.

True or False?

(a) $B \subseteq A$ ______________

(b) $A \times B = B \times A$ ______________

(c) $C \subseteq A$ ______________

(d) $(4, 4) \in C \times D$ ______________

(e) $A \cap B \cap C = \{2\}$ ______________

(f) Set D has 4 subsets. ______________

(g) $\phi \subseteq B$ ______________

(h) $B \subseteq B$ ______________

(i) $A \cup B = A$ ______________

(j) $D \subseteq A$ ______________

10. In each case, explain why set A is NOT a subset of set B.

(a) $A = \{0, 3\}$ and $B = \{1, 2, 3, 4\}$

The element 0 belongs to set A but not to set B.

(b) $A = \{\text{all red-haired people}\}$ and $B = \{\text{all women}\}$

____

____

(c) $A = \{4, 5, 6, \ldots\}$ and $B = \{4, 5, 6, 7, 8\}$

____

____

(d) $A = \{x \mid 5 < x < 8, x \in N\}$ and $B = \{2, 4, 6, 8, 10\}$

____

____

Topic Two: The Set of Whole Numbers and its Properties

On page 3 of this lesson, you were introduced to the set of natural numbers

$$N = \{1, 2, 3, \ldots\}.$$

The number zero was not invented until many centuries after the natural numbers. Such a number was needed to convey the idea of "nothingness". Also a symbol for "no units" was needed in writing numerals. Originally, blank spaces were used to indicate no ones, no tens, and so on. But blank spaces led to confusion so they were eventually replaced by the symbol 0. For example, the numeral 305 represents 3 hundreds, no tens, and 5 ones.

When we include zero along with the natural numbers, we have the set of whole numbers. The infinite set of whole numbers is tabulated as follows:

$$W = \{0, 1, 2, \ldots\}$$

For convenience, we always represent this set by the capital letter, W.

Note that the only difference between set N and set W is the inclusion of zero in set W.

A. The Whole Number Zero

Zero has some unique properties with respect to the operations of addition, subtraction, multiplication, and division.

I. Addition and Subtraction

Adding zero to any whole number or subtracting zero from any whole number leaves the number unchanged.

For any whole number a,

$$a + 0 = a$$
$$a - 0 = a$$

EXAMPLES:

NUMBER IS UNCHANGED.

$$7 + 0 = 7$$
$$9 - 0 = 9$$

Thus,

13 + 0 = ______

______ - 0 = 2

8 - ______ = 8

12 - 0 = ______

______ + 5 = 5

0 + ______ = 3

Fill in the blanks.

II. Multiplication

Multiplying any whole number by zero gives a product of zero.

For any whole number a,

$$a \times 0 = 0$$

EXAMPLES: $4 \times 0 = 0$ ← PRODUCT IS 0.

$0 \times 8 = 0$

In general, whenever zero appears as a factor the product is zero.

EXAMPLES: $6 \times 8 \times \overset{\text{ZERO IS A FACTOR.}}{0} \times 3 = 0$ ← PRODUCT IS ZERO.

$2 \times 3 \times 0 = 0$

Thus,

$12 \times 0 = \underline{\qquad}$ $\qquad 6 \times \underline{\qquad} \times 4 \times 9 = 0$

$\underline{\qquad} \times 3 = 0$ $\qquad 3 \times 8 \times 0 = \underline{\qquad}$

$0 \times \underline{\qquad} = 0$ $\qquad 3 \times 4 \times 6 \times \underline{\qquad} = 0$

Fill in the blanks.

III. Division

Whenever zero appears as the dividend (number you are dividing into) in a division question or as the numerator of a fraction, the result is zero.

For any non-zero whole number a,

$$0 \div a = 0 \quad \text{(or)} \quad \frac{0}{a} = 0, \; a \neq 0$$

EXAMPLES: $\overset{\text{DIVIDEND IS ZERO.}}{0} \div 5 = 0$

NUMERATOR IS ZERO.→ $\frac{0}{7} = 0$

Thus,

$\frac{0}{8} = \underline{\qquad}$ $\qquad \underline{\qquad} \div 9 = 0$

$\frac{\quad}{2} = 0$ $\qquad 0 \div 7 = \underline{\qquad}$

$\frac{0}{\quad} = 0$ $\qquad 0 \div \underline{\qquad} = 0$

Fill in the blanks.

Whenever zero appears as the divisor (number you are dividing by) in a division question or as the denominator of a fraction, the result is undefined.

For any whole number a,
$a \div 0$ and $\frac{a}{0}$ are undefined.

DIVISOR IS ZERO.

EXAMPLES: $5 \div 0$ is undefined.

$\frac{7}{0}$ is undefined.

DENOMINATOR IS ZERO.

Thus,

$\frac{3}{0}$ = undefined

$\frac{\_\_}{0}$ = undefined

$\frac{6}{\_\_}$ = undefined

$8 \div 0$ = ________

____ $\div 0$ = undefined

$4 \div$ ____ = undefined

Fill in the blanks.

Self-correcting Exercise #9

Answers may be found on page 47 of this lesson.

1. Tell whether each of the following statements is true or false. Give a reason for your answer.

(a) $7 \times 0 = 7$ false

We know that $a \times 0 = 0$

(b) $5 + 0 = 0$ ________

____

(c) $0 \div 9 = 0$ ________

____

(d) $25 \div 0 = 0$ ________

____

(e) $3 - 0 = 3$ ________

____

(f) $\frac{0}{6}$ is undefined. ________

____

(g) $0 \times 6 = 0$ ________

____

(h) $\frac{6}{0} = 0$ ________

____

(i) $2 \times 3 \times 0 \times 5 = 30$ ________

____

2. Simplify each expression. (Some are undefined.)

(a) $0 \div 1 =$ ______

(b) $5 \times 0 \times 8 =$ ______

(c) $0 + 11 =$ ______

(d) $\frac{12}{0} =$ ______

(e) $27 - 0 =$ ______

(f) $\frac{0}{12} =$ ______

(g) $32 \times 0 =$ ______

(h) $8 \div 0 =$ ______

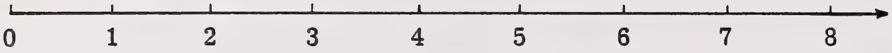
(i) $\frac{3 + 0}{2 \times 0} =$

(j) $\frac{5 - 5}{0 + 1} =$

(k) $\frac{2 \times 0}{14 - 0} =$

B. The Whole Number Line

The set of whole numbers may be associated with the points on a straight line. If a line is divided into equal segments, we can associate whole numbers with the ends of these segments.



The whole number that is paired with a given point on the number line is called the COORDINATE (ko-or′-din-it) of the point.

The arrowhead on the right of the number line indicates that it extends indefinitely to the right. Remember that 0 is the smallest whole number, but there is no largest whole number since W is an infinite set.

In dealing with any two whole numbers "a" and "b", we find that exactly one of the following three possibilities must be true.

1. $a = b$ (The numbers are equal.)
2. $a > b$ ("a" is greater than "b".)
3. $a < b$ ("a" is less than "b".)

This is called the TRICHOTOMY PROPERTY. ("trichotomy" means "divided into three parts".)

We can tell how two numbers are ordered by their positions on the number line. Numbers named further to the right are larger.

EXAMPLES:
$3 < 5$ (since 3 lies to the left of 5.)
$8 > 2$ (since 8 lies to the right of 2.)

Insert the symbol > or < to make the following statements true.

1. 12 __<__ 13
2. 1 _____ 0
3. 110 _____ 120
4. 20 _____ 200
5. 1001 _____ 1000
6. 569 _____ 568

C. Properties of Whole Numbers Under Addition and Multiplication

Addition is the operation that assigns to two numbers a third number called the SUM. The two numbers that are added are called ADDENDS or TERMS of the sum.

EXAMPLE:

$$\underbrace{3 + 4}_{\text{ADDENDS OR TERMS}} = \underset{\text{SUM}}{7}$$

Multiplication is the operation that assigns to two numbers a third number called the PRODUCT. The two numbers that are multiplied are called FACTORS of the product.

EXAMPLE: $\underbrace{2 \times 6}_{\text{FACTORS}} = \underset{\uparrow\text{PRODUCT}}{12}$

The set of whole numbers satisfies some important properties with respect to the operations of addition and multiplication. You have already accepted most of these properties intuitively. Now we will formally state them and attach names to them.

I. Closure Properties

Find the sum of each the following pairs of whole numbers.

(i) $7 + 3 =$ ____ (ii) $15 + 8 =$ ____ (iii) $4 + 7 =$ ____ (iv) $3 + 0 =$ ____

In each case, note that the result is whole number. In general, the sum of any two whole numbers is a whole number. The set of whole numbers is said to be CLOSED under the operation of addition.

CLOSURE PROPERTY OF ADDITION

For any two whole numbers a and b, $(a + b) \in W$.

Find the product of each of the following pairs of whole numbers.

(i) $7 \times 3 =$ ____ (ii) $15 \times 8 =$ ____ (iii) $1 \times 9 =$ ____ (iv) $0 \times 5 =$ ____

In each case, note that the result is a whole number. In general, the product of any two whole numbers is a whole number. The set of whole numbers is said to be CLOSED under the operation of multiplication.

CLOSURE PROPERTY OF MULTIPLICATION

For any two whole numbers a and b,

$(a \times b) \in W$.

II. Commutative Properties

The commutative property of addition tells us that when any two whole numbers are added to obtain a sum, the order in which they are added is immaterial.

> COMMUTATIVE PROPERTY OF ADDITION
>
> For any two whole numbers a and b,
> $a + b = b + a$.

EXAMPLES:

$$\left.\begin{aligned} 2 + 3 &= 3 + 2 \\ 7(m + n) &= 7(n + m) \\ 4 + (9 + 7) &= 4 + (7 + 9) \end{aligned}\right\}$$

In each case, adjacent addends were switched without affecting the sum.

Thus,

$6 + \underline{\qquad} = 9 + 6$

$2 + 12 = \underline{\qquad} + 2$

$r + s = s + \underline{\qquad}$

$$\left.\begin{aligned} 2 + (6 + 3) &= 2 + (3 + \underline{\qquad}) \\ (a + b)(c + d) &= (b + \underline{\quad})(d + \underline{\quad}) \\ 3x + 2y &= 2y + \underline{\qquad} \end{aligned}\right\}$$

Fill in the blanks.

The commutative property of multiplication tells us that when any two whole numbers are multiplied to obtain a product, the order in which they are multiplied is immaterial.

> COMMUTATIVE PROPERTY OF MULTIPLICATION
>
> For any two whole numbers a and b,
> $a \times b = b \times a$.

EXAMPLES:

$$\left.\begin{aligned} 5 \times 8 &= 8 \times 5 \\ 3(xy) &= 3(yx) \\ (a + b)(c + d) &= (c + d)(a + b) \end{aligned}\right\}$$

In each case, adjacent factors were switched without affecting the product.

Thus,

$3 \times \underline{\qquad} = 4 \times 3$

$4 \times m = \underline{\qquad} \times 4$

$3 + (2 \times 7) = 3 + (7 \times \underline{\quad})$

$$\left.\begin{aligned} r(s + t) &= (s + t)\,\underline{\quad} \\ ab + cd &= b\underline{\quad} + \underline{\quad}c \\ 3 \times (2 \times 6) &= 3 \times (\underline{\quad} \times 2) \end{aligned}\right\}$$

Fill in the blanks.

III. Associative Properties

Since addition and multiplication are binary operations, we can add or multiply only two numbers at a time. If we are asked to add or multiply three numbers, we must decide which two numbers we are going to work with first.

The associative property of addition tells us that in adding any three whole numbers, the manner in which we group the terms does not affect the sum.

ASSOCIATIVE PROPERTY OF ADDITION

For any three whole numbers a, b, and c,

$$a + (b + c) = (a + b) + c.$$

EXAMPLE: $5 + (3 + 2) = (5 + 3) + 2$

We get the same result when we add the last two numbers first as we do when we add the first two numbers first.

Thus,

$$\left.\begin{array}{l} 4 + (7 + 9) = (4 + \_\_\_) + 9 \\ (x + y) + 3z = x + (y + \_\_\_) \\ (2a + \_\_\_) + 5c = \_\_\_ + (3b + 5c) \end{array}\right\} \text{Fill in the blanks.}$$

The associative property of multiplication tells us that in multiplying any three whole numbers, the manner in which we group the factors does not affect the product.

THE ASSOCIATIVE PROPERTY OF MULTIPLICATION

For any three whole numbers a, b, and c,

$$a \times (b \times c) = (a \times b) \times c.$$

EXAMPLE: $2 \times (4 \times 6) = (2 \times 4) \times 6$

We get the same result when we multiply the adjacent factors 4 and 6 first as we do when we multiply the adjacent factors 2 and 4 first.

Thus,

$$\left.\begin{array}{l}(8 \times 3) \times 4 = 8 \times (\_\_\_ \times 4) \\[1em] 4 \times (x \times \_\_\_) = (4 \times \_\_\_) \times y \\[1em] (2 \times \_\_\_) \times 5 = 2 \times (3 \times \_\_\_)\end{array}\right\} \text{Fill in the blanks.}$$

The commutative and associative properties combined allow us to arrange the terms of a sum or factors of a product in any order so that the calculation can be done most simply. For example, the product $(4 \times 9) \times 25$ could be calculated mentally if the factors 4 and 25 were grouped together first.

i.e. $(4 \times 9) \times 25$

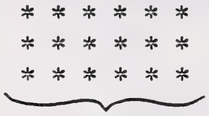
The commutative property allows us to change the order of multiplying 4 and 9.

$= (9 \times 4) \times 25$

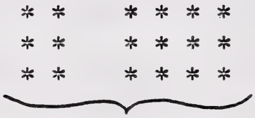
The associative property allows us to change the grouping of the factors.

$= 9 \times (4 \times 25)$
$= 9 \times 100$
$= 900$

IV. The Distributive Property



$3 \times 6 = 18$



$3(2 + 4) = (3 \times 2) + (3 \times 4) = 18$

The array of stars on the left is a pictorial representation of the product 3×6. The array on the right has the same number of stars but is subdivided vertically into two partial products (3×2) and (3×4). From the diagram, we can see that:

$$3(2 + 4) = (3 \times 2) + (3 \times 4)$$

The distributive property tells us how we can distribute a multiplier over a series of addends. In the example above, note that the product of the factors 3 and $(2 + 4)$ can be determined by distributing the factor 3 over the two addends 2 and 4.

DISTRIBUTIVE PROPERTY

For any three whole numbers a, b, and c,
$a(b + c) = ab + ac$.

We use the distributive property to multiply numbers that have more than one digit.

EXAMPLE: 3 × 52 = ▨

Solution

We must break the 2-digit number into tens and ones.

$$3 \times 52 = 3(50 + 2)$$

Then we use the distributive property to multiply the two addends, 50 and 2, by the common multiplier, 3.

$$= (3 \times 50) + (3 \times 2)$$
$$= 150 + 6$$
$$= 156$$

Use the distributive property to find the products below.

1. 8 × 65 = 8(60 + ___) = (8 × 60) + (___ × ___) = ___ + ___ = ___

2. 7 × 135 = 7(100 + 30 + ___) = (7 × ___) + (7 × ___) + (7 × ___)

= ___ + ___ + ___

= ___

The distributive property can be used in reverse to add like algebraic terms.

EXAMPLE: Simplify the sum 2y + 5y.

Solution

We cannot really add letter expressions; we can only add numbers.

We can apply the distributive property in reverse and take the common factor y out of each term in the sum.

i.e. $2y + 5y = (2 + 5)y$

Then we can add the numbers 2 and 5.

$$= 7y$$

Use the distributive property to find the following sums.

1. $6x + 5x = (\_\_\_\_\_ + \_\_\_\_\_)x = \_\_\_\_\_x$

2. $10ab + 4ab = (\_\_\_\_\_ + \_\_\_\_\_)ab = \_\_\_\_\_\_\_\_\_\_$

3. $9m + m = (\_\_\_\_\_ + 1)m = \_\_\_\_\_m$

4. $a^2 + 2a^2 = (\_\_\_\_\_ + \_\_\_\_\_)a^2 = \_\_\_\_\_\_\_\_\_\_$

V. Identity Elements

If zero is added to any whole number, that number is left unchanged. Thus, zero is called the IDENTITY ELEMENT FOR ADDITION or the ADDITIVE IDENTITY.

ADDITIVE IDENTITY PROPERTY

For any whole number a,
$a + 0 = a$.

Thus,

$$\left.\begin{array}{l} 9 + 0 = \_\_\_\_\_\_ \\ 0 + \_\_\_\_\_\_ = 6 \\ 2 + \_\_\_\_\_\_ = 2 \end{array}\right\} \text{Fill in the blanks.}$$

If any whole number is multiplied by one, that number is left unchanged. Thus, one is called the IDENTITY ELEMENT FOR MULTIPLICATION or the MULTIPLICATIVE IDENTITY.

MULTIPLICATIVE IDENTITY PROPERTY

For any whole number a,
$a \times 1 = a$.

Thus,

$$\left.\begin{array}{l} 3 \times 1 = \_\_\_\_\_ \\ 1 \times \_\_\_\_\_ = 8 \\ \_\_\_\_\_ \times 9 = 9 \end{array}\right\} \text{Fill in the blanks.}$$

The properties of the whole numbers under the operations of addition and multiplication are summarized in the chart below.

For any whole numbers a, b, and c:	Addition	Multiplication
Closure Properties	$(a + b)$ is a whole number.	$(a \times b)$ is a whole number.
Commutative Properties	$a + b = b + a$	$a \times b = b \times a$
Associative Properties	$a + (b + c) = (a + b) + c$	$a \times (b \times c) = (a \times b) \times c$
Distributive Property	$a(b + c) = (a \times b) + (a \times c)$	
Identity Elements	Additive identity is zero. $a + 0 = a$	Multiplicative identity is one. $a \times 1 = a$

Self-Correcting Exercise #10

Answers may be found on page 48 of this lesson.

1. Name the property which justifies each statement.

   (a) $5bc = 5cb$ commutative property of multiplication

   (b) $2r + t = t + 2r$ ______

   (c) $3 + (7 + 9) = (3 + 7) + 9$ ______

   (d) $5(c + d) = 5(d + c)$ ______

   (e) $2(m + n) = (m + n)2$ ______

   (f) $1 \times bc = bc$ ______

   (g) $3(ab) = (3a)b$ ______

   (h) $2(y + 1) = 2y + 2$ ______

   (i) $(c + a) + 0 = c + a$ ______

EXERCISE - Whole Numbers and Their Properties

1. Fill in the blanks.

(a) When any whole number is multiplied by zero, the result is

________________.

(b) For any whole numbers m and n, $m + n = n + m$ because

addition is ________________.

(c) The ________________ and ________________ properties of addition allow us to write $98 + (36 + 2)$ as $(98 + 2) + 36$.

(d) The ________________ property of ________________ tells us that we can change the manner of grouping three factors without affecting the product.

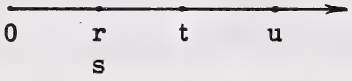
(e) The ________________ property of ________________ tells us that the sum of two whole numbers is always a whole number.

(f) Since a number is unchanged when zero is added to it, zero is

called the ________________ identity in set W.

(g) We use the ________________ property when we multiply numbers with more than one digit.

2.



Use the number line above to decide if the following statements are true or false.

(a) $u > r$ true

(b) $r > 0$ ________

(c) $s > t$ ________

(d) $u < t < s$ ________

(e) $0 < r < u$ ________

(f) $r = s$ ________

(g) $\boxed{s \leqq r} < t$ ________

This part is true either if $s < r$ or if $s = r$.

(h) $0 < u < t$ ________

(i) $t \geqq r$ ________

3. Is set N a subset of set W? __________ Why? ______________________

____

4. Decide whether the following sets are closed under the given operations.

(a) Is set O = {1, 3, 5, ...} closed under the operation of multiplication?
Solution

Multiply pairs of elements from set O and decide if the results belong to O. (i.e. are the results always odd numbers?)

1 × 3 = ______ 3 × 7 = ______ } Are the results
3 × 5 = ______ 5 × 11 = ______ } odd numbers?

______

Is set O closed under multiplication? __________

(b) Is set O = {1, 3, 5, ...} closed under the operation of addition?
Solution

Add pairs of elements from set O and decide if the results belong to O. (i.e. are the results always odd numbers?)

5 + 7 = ______ ___ + ___ = ___ } Are the results
9 + 15 = ______ ___ + ___ = ___ } odd numbers?

______

Is set O closed under addition? __________

(c) Is set E = {2, 4, 6, ...} closed under multiplication?
Solution

Multiply pairs of elements from set E and decide if the results belong to E. (i.e. are the results even numbers?)

6 × 8 = ______ ___ × ___ = ___ } Are the results
12 × 10 = ______ ___ × ___ = ___ } even numbers?

______

Is set E closed under multiplication? __________

5. Name the property which justifies each statement. Choose from the following list.

closure property of addition
closure property of mult.
commutative property of addition
commutative property of mult.
associative property of addition
associative property of mult.
additive identity property
multiplicative identity property
distributive property

(a) $(2 \times 3) \in W$ closure property of multiplication

(b) $(5 \times 7) \times 8 = 5 \times (7 \times 8)$ ________________

(c) $x \times y = y \times x$ ________________

(d) $4 + 0 = 4$ ________________

(e) $5a + 2b = 2b + 5a$ ________________

(f) $1 \times 3abc = 3abc$ ________________

(g) $(5 + 9) \in W$ ________________

(h) $2r + (3s + 2t) = (2r + 3s) + 2t$ ________________

(i) $4(a + b) = 4(b + a)$ ________________

(j) $4(a + b) = 4a + 4b$ ________________

(k) $4(a + b) = (a + b)4$ ________________

(l) $(2m + 0) + n = 2m + n$ ________________

(m) $5mn = 5nm$ ________________

(n) $(2 + 4) + 6 = 2 + (4 + 6)$ ________________

6. Simplify each expression. (Some are undefined.)

(a) $1 \times 83 =$ ________

(b) $0 \times 12 =$ ________

(c) $7 + 0 =$ ________

(d) $2x + 9x =$ ________

(e) $\frac{0}{4} =$ ________

(f) $\frac{4}{0} =$ ________

(g) $14 - 0 =$ ________

(h) $3 \times 9 \times 0 =$ ________

(i) $3(a + b) =$ ______ + ______

(j) $y + 5y =$ ______

(k) $\frac{(0 \times 5)}{7} = \frac{\_\_}{7} =$ ______

(l) $\frac{6 + 0}{3 - 0} =$

(m) $0(3 + 5) =$

(n) $6 + \frac{10}{0} =$

(o) $(3 \times 1) + 0 =$

(p) $6 - \frac{0}{2} =$

(q) $\frac{7 \times 0}{9} =$

(r) $4m + 8m =$

Key to Self-correcting Exercises in Lesson A

Exercise #1, page 2

1. In each tabulation below, the elements could have been listed in any order.

   (a) $P = \{\text{Alberta, Saskatchewan, Manitoba}\}$

   (b) $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

   (c) $V = \{a, e, i, o, u\}$

   (d) $O = \{1, 3, 5, 7, 9\}$

2. (a) $2 \in A$ (d) $6 \in B$
   (b) $2 \notin B$ (e) $9 \notin A$
   (c) $6 \in A$ (f) $9 \in B$

Exercise #2, page 4

1. (a) $A = \{4, 5, 6, \ldots, 103\}$

   (b) $B = \{33, 35, 37, \ldots\}$

   (c) $C = \{12, 14, 16, \ldots, 86\}$

   (d) $D = \{\overset{\text{FIRST TWO-DIGIT NUMBER}}{10}, 11, 12, \ldots, \overset{\text{LAST TWO-DIGIT NUMBER}}{99}\}$

2. S is the set of all multiples of 3 between 3 and 30 inclusive.
   i.e., $S = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$

   (a) false (b) false (c) true
   (d) true (e) false (f) true

Exercise #3, page 8

1. (a) $x < 35$ (d) $\overset{\text{SMALLER NUMBER}}{7} < x \leqq \overset{\text{LARGER NUMBER}}{54}$

   (b) $12 < x < 48$ (e) $4 \leqq x \leqq 16$

   (c) $x \geqq 5$ (f) $x \leqq 6$

Exercise #3, continued

2. (a) $\{x \mid x > 6,\ x \in N\}$; $\{7, 8, 9, \ldots\}$

(b) $\{x \mid 3 \leqq x < 33,\ x \in N\}$; $\{3, 4, 5, \ldots, 32\}$

(c) $\{x \mid x \leqq 30,\ x \in N\}$; $\{1, 2, 3, \ldots, 30\}$

(d) $\{x \mid 4 < x < 82,\ x \in N\}$; $\{5, 6, 7, \ldots, 81\}$

Exercise #4, page 10

1. (a) finite (68 members)
   (b) infinite
   (c) finite (1 member)
   (d) null (no members)
   (e) finite
   (f) finite (11 members)
   (g) null (no members)
   (h) infinite

Exercise #5, page 13

1. R is the set of even numbers between 2 and 32 inclusive.
   i.e. $R = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32\}$

(a) yes
(Every element belongs to set R.)

(b) no
$7 \notin R$, $9 \notin R$

(c) yes
2 belongs to set R.

(d) yes
ϕ is a subset of every set.)

(e) yes
(Every element belongs to set R.)

(f) no
(Some of these numbers aren't even.)

(g) yes
(Every set is a subset of itself.)

(h) no
($0 \notin R$)

(i) yes
(Every element belongs to set R.)

2. (a) $2^5 = 32$ (since the set has 5 elements.)
   (b) $2^4 = 16$ (since the set has 4 elements.)
   (c) $2^1 = 2$ (since the set has 1 element .)
   (d) $2^2 = 4$ (since the set has 2 elements.)
   (e) $2^3 = 8$ (since the set has 3 elements.)

3. The eight subsets are:

{ }, {blue}, {green}, {red}, {blue, green}, {blue, red},
{green, red}, {blue, green, red}

Exercise #6, page 16

1. (a) A and B have the elements 7 and 9 in common.
$A \cap B = \{7, 9\}$

(b) R and S have the elements 12, 14, 16 and 18 in common.
$R \cap S = \{12, 14, 16, 18\}$

(c) M and N have no elements in common.
$M \cap N = \{\ \}$ or $\emptyset$

(d) J and K have the elements 4, 8, and 12 in common.
$J \cap K = \{4, 8, 12\}$

(e) A, B, and C have the elements 2 and 4 in common.
$A \cap B \cap C = \{2, 4\}$

Exercise #7, page 17

1. (a) Set A contributes 6, 7, 8, and 9. Set B contributes the new element 13.
$A \cup B = \{6, 7, 8, 9, 13\}$

(b) Set M contributes 4, 5, and 6. Set N contributes the new elements 9 and 10.
$M \cup N = \{4, 5, 6, 9, 10\}$

(c) Set R contributes 4, 5, and 6. Set S contributes the new elements 1, 3, 8, and 9.
$R \cup S = \{1, 3, 4, 5, 6, 8, 9\}$

(d) Set P contributes 1, 2, 3, . . . , 15. Set Q contributes the new elements 16, 17, and 18.
$P \cup Q = \{1, 2, 3, \ldots, 18\}$

2. Set A contributes b, c, and d. Set B contributes the new elements a, f, and g. Set C contributes the new elements i and ℓ.

$$A \cup B \cup C = \{a, b, c, d, f, g, i, \ell\}$$

Exercise #8, page 20

1. (a) $A \times B = \{(1, 5), (1, 6), (3, 5), (3, 6), (5, 5), (5, 6)\}$

(b) $B \times A = \{(5, 1), (5, 3), (5, 5), (6, 1), (6, 3), (6, 5)\}$

(c) $A \times A = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

(d) $B \times B = \{(5, 5), (5, 6), (6, 5), (6, 6)\}$

Exercise #8, continued

2. $R \times S = \{(1,3),(1,4),(1,5),(3,3),(3,4),(3,5)$
$(5,3),(5,4),(5,5),(7,3),(7,4),(7,5)\}$

(a) yes
($1 \in R$ and $3 \in S$)

(b) no
(Belongs to $S \times R$)

(c) no
($2 \notin S$)

(d) yes
($3 \in R$ and $3 \in S$)

(e) no
($1 \notin S$)

(f) yes
($7 \in R$ and $4 \in S$)

(g) yes
($5 \in R$ and $3 \in S$)

(h) no
($7 \notin S$)

(i) no
($2 \notin R$)

Exercise #9, page 30

1. (a) false; we know that $a \times 0 = 0$.

(b) false; we know that $a + 0 = a$.

(c) true; we know that $0 \div a = 0$.

(d) false; we know that $a \div 0$ is undefined.

(e) true; we know that $a - 0 = a$.

(f) false; we know that $\frac{0}{a} = 0$.

(g) true; we know that $0 \times a = 0$.

(h) false; we know that $\frac{a}{0}$ is undefined.

(i) false; we know that $0 \times a = 0$.

2. (a) $0 \div 1 = \underline{0}$

(b) $5 \times 0 \times 8 = \underline{0}$

(c) $0 + 11 = \underline{11}$

(d) $\frac{12}{0}$ is undefined.

(e) $27 - 0 = \underline{27}$

(f) $\frac{0}{12} = \underline{0}$

(g) $32 \times 0 = \underline{0}$

(h) $8 \div 0$ is undefined.

(i) $\frac{3 + 0}{2 \times 0} = \frac{3}{0}$ is undefined.

(j) $\frac{5 - 5}{0 + 1} = \frac{0}{1} = \underline{0}$

(k) $\frac{2 \times 0}{14 - 0} = \frac{0}{14} = \underline{0}$

Exercise #10, page 39

1. (a) commutative property of multiplication (Factors b and c have been switched.)

   (b) commutative property of addition (Addends 2r and t have been switched.)

   (c) associative property of addition (Addends are grouped differently.)

   (d) commutative property of addition (Addends c and d have been switched.)

   (e) commutative property of multiplication (Factors 2 and (m + n) have been switched.)

   (f) multiplicative identity property (Multiplicative identity is 1.)

   (g) associative property of multiplication (Factors are grouped differently.)

   (h) distributive property (Factor 2 is distributed over the addends y and 1.)

   (i) additive identity property (Additive identity is 0.)

End of Lesson A

Lesson B

THE SET OF REAL NUMBERS AND ITS SUBSETS

Topic One: The Set of Integers

In Lesson A, you learned that the set of whole numbers, W, is closed under the operations of addition and multiplication. This means that when we add or multiply two whole numbers, the result is always a whole number.

Is set W closed under subtraction? Note that we can find a whole number that satisfies each of the conditions given in List 1 below, but no whole numbers exist that will satisfy the conditions given in List 2.

List 1	List 2
9 - 8 = ■	8 - 9 = ■
7 - 4 = ■	4 - 7 = ■
14 - 3 = ■	3 - 14 = ■
5 - 5 = ■	

In general, we find that it is possible to perform a subtraction in set W if and only if the number from which you subtract (the minuend) is larger than the number to be subtracted (the subtrahend). Since the difference of two whole numbers is not always a whole number, set W is not closed under the operation of subtraction.

The SET OF INTEGERS was invented in order to give closure under subtraction. It also provides us with numbers that can be used to show opposites such as profit and loss.

A. Definition of Integers

The set of integers contains elements which are less than zero, besides those elements which already belong to set W. The elements which are less than zero are obtained by placing minus signs in front of the natural numbers 1, 2, 3, ... Thus, for every natural number "a", we can write a new number "-a" (read "negative a") which is less than zero. These new numbers -1, -2, -3, ... that are less than zero are called NEGATIVE INTEGERS. The familiar natural numbers 1, 2, 3, ... can be called POSITIVE INTEGERS.

If we take the union of the set of negative integers, the set of positive integers, and the set containing the one element zero, we obtain the set of integers.

$$\text{i.e.}\quad \{\text{all integers}\} = \{\text{all neg. integers}\} \cup \{\text{all pos. integers}\} \cup \{0\}$$
$$= \{-1, -2, -3, \ldots\} \cup \{1, 2, 3, \ldots\} \cup \{0\}$$

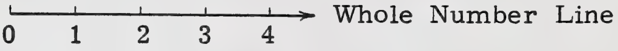
The set of integers is always represented by the capital letter I and is tabulated as follows:

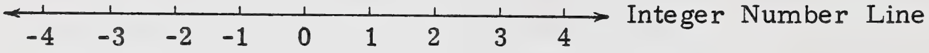
$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The three dots that appear at the beginning and end of the tabulation indicate that this is an infinite set that has no least nor greatest element.

B. The Integer Number Line

If we extend the whole number line to the left we can represent the set of integers graphically. Points corresponding to the negative integers can be located to the left of zero.





The arrowheads on both ends of the integer number line indicate that it extends indefinitely in two directions, unlike the whole number line which extends indefinitely in only one direction. The point corresponding to zero on the number line is called the ORIGIN.

The numbers are ordered on the integer number line in the same way they are ordered on the whole number line.

For any two integers "a" and "b", if point "a" lies to the right of point "b" on the number line then "a" is greater than "b" and we can write $a > b$.

EXAMPLES:

$2 > -2$ since 2 lies to the right of -2 on the number line.
$-3 > -4$ since -3 lies to the right of -4 on the number line.

On the other hand, if point "a" lies to the left of point "b" on the number line, then "a" is less than "b" and we write $a < b$.

EXAMPLES:

$-4 < 0$ since -4 lies to the left of 0 on the number line.
$-2 < -1$ since -2 lies to the left of -1 on the number line.

We can think of all integers, except zero, as possessing both magnitude and direction. The distance between a number and the origin on the number line is called the MAGNITUDE or ABSOLUTE VALUE of the number. The direction of a number is determined by its sign. For example, the integer 2 has a magnitude of 2 since it is a distance of 2 units from the origin. It has a positive direction and lies to the right of the origin. The integer -2 also has a magnitude of 2 since it is a distance of 2 units from the origin. But it has negative direction and lies to the left of the origin.

The absolute value of an integer is indicated by placing the integer between vertical bars. Thus, for any integer "a", the symbol $|a|$ represents the absolute value of "a".

EXAMPLES:

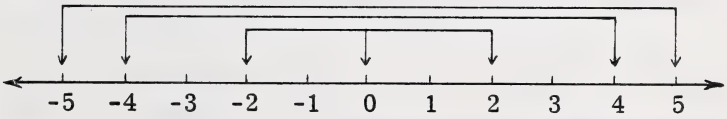
1. The absolute value of -3 is 3. We can write $|-3| = 3$. This is read, "the absolute value of negative 3 is 3."

2. The absolute value of 7 is 7. We can write $|7| = 7$. This is read, "the absolute value of 7 is 7."

Thus,
$|6|$ = ______, $|-5|$ = ______, $|0|$ = ______, $|-12|$ = ______

ADDITIVE INVERSES

Two integers which have the same absolute value but opposite direction are called ADDITIVE INVERSES of each other. Every integer, except zero, has as its additive inverse a number on the other side of the origin at an equal distance from the origin. The additive inverse of zero is zero.



EXAMPLES:

1. -5 and 5 are additive inverses.

2. -4 and 4 are additive inverses.

What is the additive inverse of -3? ______ of 6? ______ of 1? ______

Self-correcting Exercise #1

Answers may be found on page 58 of this lesson.

1. Fill in the blanks in the chart below. (First decide if "a" lies to the right or left of "b". Then state if "a" is greater than or less than "b".)

	a	b	Right or Left?	a > b or a < b?
(a)	-4	-1	-4 lies to the left of -1.	-4 < -1
(b)	0	-9	0 lies to the _____ of -9.	0 _____ -9
(c)	-6	-8	-6 lies to the _____ of -8.	-6 _____ -8
(d)	12	5	12 lies to the _____ of 5.	12 _____ 5
(e)	-3	1	-3 lies to the _____ of 1.	-3 _____ 1

2. Use an integer to represent each of the following situations.

(a) 10° below zero

-10

(b) 30° above zero

30

(c) a loss of 5 yards

_____

(d) 50 years ago

_____

(e) a balance of $20

_____

(f) no gain or loss

_____

(g) a raise of $100

_____

(h) a debt of $15

_____

3. Give the absolute value of each integer.

(a) $|-11| =$ _____ (b) $|0| =$ _____ (c) $|8| =$ _____ (d) $|-1| =$ _____

4. Give the additive inverse of each integer.
   (a) The additive inverse of 14 is _______.
   (b) The additive inverse of -50 is _______.
   (c) The additive inverse of 1 is _______.
   (d) The additive inverse of -15 is _______.

C. Operating With Integers

Much of your future work in algebra will depend on your ability to operate with integers. At this point, we will review the rules for multiplying, dividing, adding, and subtracting integers.

I. Multiplying Integers

> The product of two integers with like signs is always positive.

Thus, the product of two positive integers or two negative integers is always positive.

EXAMPLES:	Find these products.	
2 × 3 = 6	7 × 2 = _______	4 × 2 = _______
5 × 2 = 10	-9 × -9 = _______	-4 × -2 = _______
-2 × -3 = 6	-10 × -1 = _______	3 × 4 = _______
-5 × -2 = 10	positive product	

> The product of two integers with unlike signs is always negative.

Thus, if only one of the two factors in a product is negative, the product will be negative.

EXAMPLES:	Find these products.	
5 × -7 = -35	-5 × 8 = _______	7 × -3 = _______
-8 × 3 = -24	4 × -2 = _______	-2 × 2 = _______
9 × -1 = -9	-3 × 1 = _______	5 × -1 = _______
	negative product	

Fill in the blanks in the following chart.

	Are the signs of the factors alike or different?	Is the product positive or negative?	Product
-3 × 7	different	negative	-21
-2 × -8			
2 × 9			
6 × -8			
-3 × -3			
-3 × 11			
-1 × 6			
-1 × -1			

The rules for multiplying integers can be extended to cover any number of factors.

1. An odd number of negative factors produces a negative product.

   EXAMPLE:
   $-2 \times 3 \times 5 \times -7 \times -1 = -210 \leftarrow$ NEGATIVE PRODUCT

   There are three negative factors. Since 3 is an odd number, the product is negative.

2. An even number of negative factors produces a positive product.

   EXAMPLE:
   $-1 \times 2 \times -1 \times 3 \times -1 \times 4 \times -1 = 24 \leftarrow$ POSITIVE PRODUCT

   There are four negative factors. Since 4 is an even number, the product is positive.

Fill in the blanks in the following chart.

	Even or odd number of negative factors?	Is the product pos. or neg.?	Product
2 × -3 × 4 × -2	even	positive	48
-1 × -1 × -1 × -1 × -1 × -1			
-1 × -2 × -3 × -4 × -5			
6 × 3 × -2 × 1			
-1 used as a factor 7 times			
-1 used as a factor 10 times			
-3 × -3 × -3 × -3			
-5 × -2 × 3 × 2			

II. Dividing Integers

The same sign rules that were used for multiplying two integers can be applied to the quotient of two integers.

> The quotient of two integers with like signs is always positive.

Thus, the quotient of two positive integers or two negative integers is always positive.

EXAMPLES:

$8 \div 2 = 4$ (or $\frac{8}{2} = 4$)

$-16 \div -2 = 8$ (or $\frac{-16}{-2} = 8$)

$-18 \div -9 = 2$ (or $\frac{-18}{-9} = 2$)

POSITIVE QUOTIENT

Find these quotients.

$-24 \div -6 =$ ______ $9 \div 1 =$ ______

$\frac{-35}{-7} =$ ______ $\frac{-10}{-2} =$ ______

$22 \div 2 =$ ______ $-8 \div -8 =$ ______

$\frac{28}{7} =$ ______ $\frac{24}{8} =$ ______

The quotient of two integers with unlike signs is always negative.

Thus, if only one of the numbers in a division question is negative, the quotient will be negative.

EXAMPLES:

$-8 \div 2 = -4$ (or $\frac{-8}{2} = -4$)

$16 \div -8 = -2$ (or $\frac{16}{-8} = -2$)

$-5 \div 1 = -5$ (or $\frac{-5}{1} = -5$)

NEGATIVE QUOTIENT

Find these quotients.

$14 \div -2 =$ ______

$-2 \div 2 =$ ______

$\frac{-9}{3} =$ ______

$\frac{21}{-3} =$ ______

$-36 \div 9 =$ ______

$-15 \div 3 =$ ______

$\frac{4}{-1} =$ ______

$\frac{8}{-8} =$ ______

Use the rules for dividing integers to find the following quotients.

1. $-32 \div -4 =$ ______
2. $81 \div 9 =$ ______
3. $-7 \div 1 =$ ______
4. $\frac{16}{-4} =$ ______
5. $\frac{-42}{-6} =$ ______
6. $\frac{-10}{2} =$ ______
7. $\frac{3}{-1} =$ ______
8. $63 \div -7 =$ ______
9. $\frac{-1}{-1} =$ ______

III. Adding Integers

The sum of two integers with like signs can be found by adding the absolute values and attaching the common sign.

Thus, the sum of two positive integers can be found by adding the absolute values and making the sum positive.

EXAMPLES:

$5 + 8 = 13$

$6 + 11 = 17$

$2 + 7 = 9$ ← POSITIVE SUM

Find the sums.

$5 + 7 =$ ______

$4 + 3 =$ ______

$13 + 2 =$ ______

$1 + 2 =$ ______

$9 + 1 =$ ______

$8 + 8 =$ ______

The sum of two negative integers can be found by adding the absolute values and making the sum negative.

EXAMPLES:

(-3) + (-5) = -8

(-1) + (-8) = -9

(-7) + (-4) = -11 ↖ NEGATIVE SUM

Find these sums.

(-12) + (-2) = _____   (-4) + (-4) = _____

(-6) + (-1) = _____   (-15) + (-6) = _____

(-3) + (-1) = _____   (-7) + (-6) = _____

> The sum of two integers with unlike signs can be found by subtracting the absolute values and attaching the sign of the integer with the larger absolute value.

This means that if a positive integer and a negative integer are added, the sum will be positive if the positive integer has the larger absolute value. The sum will be negative if the negative integer has the larger absolute value.

EXAMPLES:

9 + (-3) = 6

The difference of 9 and 3 is 6. The sum is positive since 9, which has the larger absolute value, is positive.

-8 + 5 = -3

The difference of 8 and 5 is 3. The sum is negative since -8, which has the larger absolute value, is negative.

> The sum of an integer and its additive inverse is zero.

EXAMPLES:

-4 + 4 = 0
3 + (-3) = 0

In each case the two numbers are additive inverses and their sum is zero.

Use the rules for adding integers with unlike signs to find the following sums.

1. 6 + (-5) = 1   2. -8 + 8 = _____   3. -9 + 5 = _____

4. -3 + 10 = _____   5. 2 + (-4) = _____   6. 5 + (-5) = _____

Fill in the blanks in the following chart.

	Are the signs alike or different?	Must you add or subtract the absolute values?	Is the result positive or negative?	Result
(-4) + (-5)	alike	add	negative	-9
5 + (-7)				
-9 + 11				
-1 + 1			neither	
2 + 3				
(-7) + (-2)				
2 + (-2)				
-7 + 6				
8 + (-5)				
(-1) + (-4)				

IV. Subtracting Integers

In order to subtract two integers:

1. Change the operation sign to addition.
2. Reverse the sign of the second number.
3. Use the rules for adding integers.

EXAMPLES:

-2 - (-5)

Change operation to addition and change -5 to 5.

NOTE SIGN CHANGES.

= -2 + (5)

Use the rule for adding two integers with unlike signs.

= 3

-6 - 8

Change operation to addition and change 8 to -8.

NOTE SIGN CHANGES.

= -6 + (-8)

Use the rule for adding two integers with like signs.

= -14

In the exercises below, change each subtraction question to an addition question. This can be done by changing the operation to addition and reversing the sign of the second number.

1. 5 - 8 = 5 + (-8) ← 8 becomes -8.
2. -9 - (-3) = ___ + 3 ← -3 becomes 3.
3. 12 - 20 = ___ + ___
4. 1 - (-4) = ___ + ___
5. -4 - 5 = ___ + ___
6. 2 - (-1) = ___ + ___
7. 2 - 9 = ___ + ___
8. -3 - 8 = ___ + ___

In the table below, write each subtraction question as an addition question. Then decide if the signs of the two integers are alike or different. In the last column give the sum.

Subtraction question	Corresponding addition question	Are the signs alike or different?	Sum
12 - (-7)	12 + 7	alike	19
-5 - 9	-5 + (-9)	___	___
-2 - (-6)	___ + ___	___	___
-7 - 8	___ + (-8)	___	___
-5 - (-1)	___ + ___	___	___
2 - 10	___ + (-10)	___	___
-3 - 6	___ + ___	___	___
8 - 9	___ + ___	___	___
4 - (-3)	___ + ___	___	___

Self-correcting Exercise #2

Answers may be found on page 58 of this lesson.

1. Use the rule for subtracting integers to find the following differences. (i.e. change the operation to addition, change the sign of the second number, and then add.)

(a) 6 - 9 = 6 + (-9) = ___
(b) 17 - (-8) = ___ + ___ = ___
(c) -13 - (-4) = ___ + ___ = ___
(d) -7 - 5 = ___ + ___ = ___
(e) -2 - (-2) = ___ + ___ = ___
(f) 3 - 6 = ___ + ___ = ___

2. Fill in the blanks in the following chart.

	Are the signs alike or different?	Will the result be positive or negative?	Result
(a) 16 ÷ -4	Different	Negative	______
(b) 4 + (-7)	______	______	______
(c) 6 × -5	______	______	______
(d) $\frac{-2}{-1}$	______	______	______
(e) (-3) + (-1)	______	______	______
(f) $\frac{14}{-2}$	______	______	______
(g) -3 × -3	______	______	______
(h) -9 + 13	______	______	______
(i) -10 ÷ -5	______	______	______

3. Simplify the following expressions.

(a) 6 × -5 = ______ (b) -8 ÷ -1 = ______

(c) -15 + 12 = ______ (d) 5 - 12 = ______

(e) -4 × -4 = ______ (f) $\frac{-21}{7}$ = ______

(g) -3 - (-15) = ______ (h) 17 - (-1) = ______

(i) (-6) + (-6) = ______ (j) -6 + 6 = ______

(k) -3 - 2 = ______ (l) $\frac{10}{-1}$ = ______

(m) -8 + 9 = ______ (n) -6 - (-4) = ______ + 4 = ______

(o) $-5 + 2 - 8 + 4 = \overset{\text{SUM OF NEGATIVES}}{-13} + \overset{\text{SUM OF POSITIVES}}{6} = \underline{\qquad}$

(p) $3 - 9 - 1 + 12 =$

(q) $-6 + 10 + 3 - 9 =$

(r) $2 - (-4) - 8 + 7 = 2 + 4 - 8 + 7 =$

(s) $\dfrac{7 - 12}{-5} =$

(t) $\dfrac{-8 - 12}{-4} =$

(u) $\dfrac{-4\,(2 - 6)}{-2} =$

(v) $-1(8 - 4 - 3 - 6) =$

D. Properties of the Set of Integers

The set of integers satisfies the same properties as the whole numbers under the operations of addition and multiplication. (Review the summary of whole number properties on page 39, lesson A.) But, the integers have a new property under the operation of addition that does not apply to the set of whole numbers. Every integer has an inverse element under the operation of addition. The sum of an integer and its additive inverse is zero.

> DEFINITION - ADDITIVE INVERSE
>
> For every integer "a", there is a unique integer "-a" such that:
>
> $$a + (-a) = 0$$
>
> The integers "a" and "-a" are additive inverses of each other.

EXAMPLES:

1. 7 and -7 are additive inverses since $7 + (-7) = 0$.

2. -3 and 3 are additive inverses since $-3 + 3 = 0$.

Remember that the additive inverse of a positive integer is always negative, the additive inverse of a negative integer is always positive, and the additive inverse of zero is zero.

Additive inverses do not exist in the set of whole numbers.

Below is a summary of the properties of set I under the operations of addition and multiplication. Fill in the blanks.

For any integers a, b, c:	Addition	Multiplication
Closure Properties	(a + b) is an integer.	(a × b) is an ______________.
Commutative Properties	a + b = b + ____	a × b = ____ × a
Associative Properties	a + (b + c) = (a + ___) + ___	a × (b × c) = (___ × ___) × c
Distributive Property	a(b + c) = _____ + ac	
Identity Elements	Additive identity is 0. a + 0 = ______	Multiplicative identity is 1. a × 1 = _______
Inverse Element	* Additive inverse of a is -a. a + (-a) = ______	

* This is a property of the integers but not of the whole numbers.

Let us check some of these properties with respect to the operation of subtraction.

1. Is set I closed under subtraction?

   Find the following differences.

   6 - 5 = _____ -4 - 5 = _____

   7 - 9 = _____ 9 - 9 = _____

   3 - (-1) = _____ -2 - (-1) = _____

   Are the results all integers? __________

   Since the difference of two integers is always an integer, set I is closed under subtraction.

2. Is subtraction commutative in set I?

   EXAMPLE: Is it true that 8 - 5 = 5 - 8?
   Solution
   8 - 5 = _____ and 5 - 8 = _____

   Are the two sides equal? __________

   Since the order in which we subtract two integers does affect the result, subtraction is not commutative in set I.

3. Is subtraction associative in set I?

   EXAMPLE: Is it true that 6 - (8 - 3) = (6 - 8) - 3?
   Solution

6 - (8 - 3)	(6 - 8) - 3
= 6 - ___	= _____ - 3
= _____	= _____

   Are the two sides equal? ________

   Since the manner in which we group three numbers in subtraction does affect the result, subtraction is not associative in set I.

4. Does multiplication distribute over subtraction in set I?

EXAMPLE: Is it true that $3(5 - 8) = (3 \times 5) - (3 \times 8)$?

Solution

$3(5 - 8)$	$(3 \times 5) - (3 \times 8)$
$= 3$ (___)	$=$ ____ $-$ ____
$=$ ____	$=$ ____

Are the two sides equal? ______

In general, a multiplier may be distributed over the terms in a difference.

i.e. $\boxed{a(b - c) = ab - ac}$

EXERCISE - The Set of Integers

1. Fill in the blanks.

(a) Numbers with minus signs designate ______________ integers and represent points to the ______________ of the origin on the number line.

(b) The integer ______________ is neither positive nor negative.

(c) -2 is ______________ than -5 since it lies to the ______________ of -5 on the number line.

(d) The symbol $|-7|$ represents the ______________ ______________ of -7 and equals ______________.

(e) The sum of an integer and its ______________ ___________ is zero.

(f) The product of two negative integers is a ______________ integer.

(g) The quotient of two integers with unlike signs is a ______________ integer.

(h) The integers ____ and ____ both have an absolute value of 3.

(i) The sum of negative integers is a ______________ integer.

(j) The product of an odd number of negative integers is a ____________ integer.

(k) The sum of two integers with unlike signs is negative if the integer with the larger absolute value is ______________.

(l) Set I is closed under ______________, while set W is not.

(m) In set I, zero is the ______________ identity and one is the ______________ identity.

(n) The operations of ______________ and ______________ are not associative nor commutative in set I.

(o) In order to subtract one integer from another, we must change the operation to ______________ and give the additive ______________ of the second integer.

(p) $3(x - y) = 3x - 3y$ because multiplication ______________ over subtraction.

2. Fill in the blanks.

(a) $-5 + (-3) =$ _____

(b) $-7 - (-10) =$ _____

(c) $-8 + 15 =$ _____

(d) $12 - 19 =$ _____

(e) $-5 \times -8 =$ _____

(f) $-18 \div 2 =$ _____

(g) $-4 - 11 =$ _____

(h) $-18 + 13 =$ _____

(i) $-1 \times 7 =$ _____

(j) $-9 \div -1 =$ _____

(k) $3 \times -13 =$ _____

(l) $\frac{25}{-5} =$ _____

(m) $-2 \times 1 \times -3 \times -1 =$ _____

(n) $-2 \times -2 \times -2 \times -2 =$ _____

(o) $\frac{-64}{-8} =$ _____ (p) $0 \times -3 =$ _____

(q) $7 + 9 - 24 =$ _____ (r) $-2 - 5 - 7 =$ _____

(s) $-10 \times -10 =$ _____ (t) $\frac{13}{-1} =$ _____

(u) $6 - 2 - 9 =$ _____ (v) $-8 + 5 + 7 =$ _____

(w) $5 \times -1 =$ _____ (x) $-1 \times -1 \times -1 \times -1 \times -1 =$ _____

(y) $-3 \times -4 \times -5 =$ _____ (z) $-1 \times 3 \times -9 =$ _____

3. Write each expression as a single integer. (Simplify within brackets first.)

(a) $(-5 \times 1) + (7 \times -1) =$ _____ + _____ = _____

(b) $-5(2 - 6) =$

(c) $\frac{-9 + (-1 \times -5)}{2} = \frac{-9 + \underline{\qquad}}{2} = \frac{\underline{\qquad}}{2} = \underline{\qquad}$

(d) $\frac{(5 - 9)}{-2} =$

(e) $\frac{(-3 + 2)(-2 - 7)}{3} =$

(f) $5 - (2 \times -3) - 15 =$

(g) $\frac{-18}{9} + 2 =$

(h) $(-4 \times 3) + (-1 \times -1) =$

(i) $\frac{5 - (-3)}{-4} =$

(j) $5 - 6 + 8 - (-4) =$

4. Fill in the blanks to make the statements true. Then name the property which justifies each statement. Choose from the following list.

- commutative prop. of addition
- commutative prop. of mult.
- associative prop. of addition
- associative prop. of mult.
- closure prop. of addition
- closure prop. of mult.
- additive identity prop.
- mult. identity prop.
- additive inverse prop.
- distributive property

	Justification
(a) 5 + _____ = 0	additive inverse prop.
(b) (-8) + 7 = _____ + (-8)	_____
(c) 4 × (-6 × 5) = (4 × -6) × _____	_____
(d) (4 × -3) ϵ _____	_____
(e) 1 × -6 = _____	_____
(f) -8 + _____ = -8	_____
(g) -2(-3 + 5) = (-2 × ___) + (-2 × ___)	_____
(h) -7 + 7 = _____	_____
(i) -3 × -9 = _____ × -3	_____
(j) (-5 + 2) ϵ _____	_____
(k) -7 + _____ = 9 + (-7)	_____
(1) 0 + (-3) = _____	_____
(m) _____ × 4 = 4	_____
(n) _____ (4 + 7) = (3 × 4) + (3 × 7)	_____
(o) (-8 + 4) + 3 = _____ + (4 + 3)	_____

Topic Two: The Set of Rational Numbers

In Topic One of this lesson, you learned that the set of integers, I, is closed under the operations of addition, subtraction, and multiplication. This means that when we add, subtract, or multiply two integers, the result is always an integer.

Is set I closed under division? Note that we can find an integer that satisfies each of the conditions given in List 1 below, but no integers exist that will satisfy the conditions given in List 2.

List 1	List 2
$8 \div -4 =$ ▨	$8 \div 5 =$ ▨
$-6 \div -3 =$ ▨	$-3 \div -6 =$ ▨
$-15 \div 3 =$ ▨	$-12 \div 9 =$ ▨

In general, we find that it is possible to perform a division in set I if and only if the number you divide by (the divisor) is a factor of the number you are dividing into (the dividend). Since the quotient of two integers is not always an integer, set I is not closed under the operation of division.

The SET OF RATIONAL NUMBERS was invented to give closure under division (except for division by zero). It also provides us with numbers that represent parts of a whole.

A. Definition of Rational Numbers

The set of rational numbers contains elements of the form

$$\frac{-6}{2}, \quad \frac{3}{-5}, \quad \frac{-4}{-2}, \quad \frac{0}{6}, \quad \frac{-4}{2}$$

that represent the quotient of two integers. In our new notation, a horizontal bar replaces the familiar division sign, "÷". For example, the rational number $\frac{-6}{2}$ represents the quotient $(-6 \div 2)$.

In general, for any two integers "a" and "b", where $b \neq 0$, there is a numeral $\frac{a}{b}$ (read "a over b") which represents the quotient $(a \div b)$. The symbol $\frac{a}{b}$ is called a FRACTION; a is the NUMERATOR, b is the DENOMINATOR, a and b are called the TERMS of the fraction. The denominator of a fraction can never be zero since division by zero is undefined. Thus, fractions like $\frac{8}{0}$, $\frac{-9}{0}$, and $\frac{32}{0}$ are undefined since their denominators are zero.

The infinite set of rational numbers cannot be tabulated in an orderly fashion so it is usually written in set-builder notation as follows:

$$Q = \{\frac{a}{b} \mid a,\ b \in I,\ b \neq 0\}$$

This is read, "Q is the set of all numerals of the form a over b such that a and b are integers and b does not equal zero." Note that, for convenience, we will always designate this set by the capital letter Q.

The set of integers is a subset of the set of rational numbers. This means that every integer is a rational number and hence can be written in the rational form $\frac{a}{b}$. All rational numbers in which the denominator divides evenly into the numerator (i.e. the denominator is a "factor" of the numerator) represent integers. For example, the rational number $\frac{-15}{3}$ represents the quotient $(-15 \div 3)$ and equals the integer -5.

What integer is each of the following rational numbers equivalent to?

1. $\frac{12}{-2}$ = -6　　2. $\frac{35}{7}$ = ______　　3. $\frac{-9}{1}$ = ______

4. $\frac{-100}{-10}$ = ______　　5. $\frac{0}{7}$ = ______　　6. $\frac{-42}{-6}$ = ______

7. $\frac{-8}{-8}$ = ______　　8. $\frac{625}{-25}$ = ______　　9. $\frac{11}{-1}$ = ______

Like integers, rational numbers are either positive, negative, or zero. You will recall that the quotient of two positive or two negative integers is always positive. Thus, any rational number that has two positive terms or two negative terms must be a positive rational number. For example, the following rational numbers are all positive:

$$\frac{-3}{-2},\quad \frac{6}{1},\quad \frac{-5}{-4},\quad \frac{9}{6},\quad \frac{-16}{-4}$$

To cut down on the number of signs of quality that appear in a rational number, numerals of the form $\frac{-a}{-b}$ (where $a,\ b \in N$) are generally written in the form $\frac{a}{b}$.

EXAMPLES:

$\frac{-3}{-2}$ is usually written as $\frac{3}{2}$.

$\frac{-1}{-5}$ is usually written as $\frac{1}{5}$.

$\frac{3}{4}$ is left unchanged.

Remember that the quotient of two integers with unlike signs is always negative. Thus, any rational number that has one positive term and one negative term must be a negative rational number. For example, the following rational numbers are all negative:

$$\frac{-3}{2} \quad \frac{4}{-5} \quad \frac{-6}{2} \quad \frac{9}{-11} \quad \frac{-12}{4}$$

To simplify notation, negative rational numbers are generally written in the form $\frac{-a}{b}$ (where a, b $\in$ N). In other words, the minus sign is generally placed in the numerator of the fraction.

EXAMPLES:

$\frac{5}{-6}$ is usually written as $\frac{-5}{6}$

$\frac{9}{-7}$ is usually written as $\frac{-9}{7}$

$\frac{-1}{4}$ is left unchanged.

Write each of the following rational numbers in the form $\frac{a}{b}$ or $\frac{-a}{b}$ where a and b are natural numbers.

1. $\frac{-7}{-8}$ ________
2. $\frac{-6}{5}$ $\frac{-6}{5}$
3. $\frac{5}{-9}$ ________
4. $\frac{-9}{-11}$ ________
5. $\frac{3}{-7}$ ________
6. $\frac{-3}{4}$ ________

B. Equal Rational Numbers

Many rational numbers name the same number. For example, the rational numbers $\frac{3}{1}$, $\frac{6}{2}$, $\frac{9}{3}$, $\frac{12}{4}$, ... all name the integer 3. If two rational numbers are equal, the cross products of their terms will be equal. For example, $\frac{3}{1} = \frac{6}{2}$ and $(3 \times 2) = (1 \times 6)$.

EQUAL RATIONAL NUMBERS

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$,

$\frac{a}{b} = \frac{c}{d}$ iff. $ad = bc$.

IF AND ONLY IF

For each pair of rational numbers below, find the cross products of the terms and decide whether or not the fractions are equal.

Fraction Pair	Cross Products	Equal or not Equal?
$\frac{-15}{20}$, $\frac{-3}{4}$	$-15 \times 4 = -60$ $20 \times -3 = -60$	equal
$\frac{3}{8}$, $\frac{12}{33}$		
$\frac{8}{36}$, $\frac{2}{9}$		
$\frac{2}{3}$, $\frac{-4}{6}$		
$\frac{-4}{10}$, $\frac{-14}{35}$		

For a given rational number, how can we obtain another rational number that is equal to it? For example, we know that $\frac{2}{3}$ and $\frac{4}{6}$ are equal rational numbers because the cross products of their terms are equal (i.e. $2 \times 6 = 3 \times 4$). We can change $\frac{2}{3}$ to the equal rational number $\frac{4}{6}$ if we multiply both its numerator and denominator by 2.

MULTIPLY BOTH TERMS BY 2.

$$\text{i.e.} \quad \frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6}$$

We can change $\frac{4}{6}$ to the equal rational number $\frac{2}{3}$ if we divide both its numerator and denominator by 2.

DIVIDE BOTH TERMS BY 2.

$$\text{i.e.} \quad \frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

In general, if the numerator and denominator of any rational number are multiplied or divided by the same integer (not zero), an equal rational number is obtained.

PRINCIPLE OF EQUAL FRACTIONS

(i) $\frac{a}{b} = \frac{ax}{bx}$

(ii) $\frac{a}{b} = \frac{a \div x}{b \div x}$

where x is a non-zero integer.

We use part (i) of the Principle of Equal Fractions to express a rational number in a form with a larger denominator. (This device must be used in adding and subtracting rational numbers with unlike denominators.) For example, the rational number $\frac{-3}{5}$ can be expressed with a denominator of 30 if both its terms are multiplied by 6.

$$\text{i.e.} \quad \frac{-3}{5} = \frac{-3 \times 6}{5 \times 6} = \frac{-18}{30}$$

Express each fraction below with the denominator as specified.

1. $\frac{-3}{4}$ in thirty-seconds $\frac{-24}{32}$ 2. $\frac{5}{4}$ in eighths _____

3. $\frac{4}{5}$ in twentieths _____ 4. $\frac{-2}{3}$ in fifteenths _____

5. $\frac{1}{4}$ in sixty-fourths _____ 6. 5 in thirds _____

We use part (ii) of the Principle of Equal Fractions to express a rational number in a form with a smaller denominator. A rational number is in LOWEST TERMS when the only numbers that will divide evenly into both the numerator and denominator are 1 and -1. In order to put a rational number in lowest terms, we must look for the largest number that will divide evenly into both terms. For example, the rational number $\frac{16}{24}$ can be put in lowest terms if we divide both its numerator and denominator by 8. (8 is the largest number that will divide evenly into both 16 and 24.)

$$\text{i.e.} \quad \frac{16}{24} = \frac{16 \div 8}{24 \div 8} = \frac{2}{3}$$

When calculating with rational numbers, you should always try to express your answers in lowest terms.

Reduce each of the following fractions to lowest terms.

1. $\frac{30}{66}$ = _____ 2. $\frac{-40}{56}$ = _____ 3. $\frac{10}{100}$ = _____

4. $\frac{14}{21}$ = _____ 5. $\frac{64}{48}$ = _____ 6. $\frac{-36}{60}$ = _____

C. Operating With Rational Numbers

If you are proficient at operating with rational numbers, this will help you considerably in operating with rational expressions. (See Lesson E of this course.) We will now briefly review the rules for multiplying, dividing, adding, and subtracting rational numbers.

I. Multiplying Rational Numbers

> The product of two rational numbers can be found by taking the product of the two numerators and dividing this result by the product of the two denominators.
>
> i.e. $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$

EXAMPLE:

PRODUCT OF NUMERATORS

$$\frac{-3}{4} \times \frac{-5}{7} = \frac{-3 \times -5}{4 \times 7} = \frac{15}{28}$$

PRODUCT OF DENOMINATORS

When multiplying fractions, we often use a short-cut known as "cancelling" to simplify our work so that the answer is in lowest terms. Cancelling involves dividing the numerators and denominators of the given fractions by the same number before completing the multiplication.

EXAMPLE: $\frac{-3}{16} \times \frac{32}{27} =$ ▨ .

Solution

$$\frac{-3}{16} \times \frac{32}{27}$$

$$= \frac{-3 \times 32}{16 \times 27}$$

Divide the -3 in the numerator and the 27 in the denominator by the common factor 3.

$$= \frac{\overset{-1}{\cancel{-3}} \times 32}{16 \times \underset{9}{\cancel{27}}} \qquad (-3 \div 3 = -1) \quad (27 \div 3 = 9)$$

Divide the 32 in the numerator and the 16 in the denominator by the common factor 16.

$$= \frac{\overset{-1}{\cancel{-3}} \times \overset{2}{\cancel{32}}}{\underset{1}{\cancel{16}} \times \underset{9}{\cancel{27}}} \qquad (32 \div 16 = 2) \quad (16 \div 16 = 1)$$

Now, the multiplication can be completed and the answer will be in lowest terms.

$$= \frac{-1 \times 2}{1 \times 9}$$

$$= \frac{-2}{9}$$

Find the following products. Use cancelling to express your answers in lowest terms.

1. $\frac{-5}{4} \times \frac{16}{25} = \quad \frac{\overset{-1}{\cancel{-5}} \times \overset{4}{\cancel{16}}}{\underset{1}{\cancel{4}} \times \underset{5}{\cancel{25}}} = \frac{-1 \times 4}{1 \times 5} = \frac{-4}{5}$

2. $\frac{6}{45} \times \frac{9}{22} =$

3. $\frac{33}{8} \times \frac{4}{15} =$

4. $\frac{-14}{15} \times \frac{-9}{56} =$

II. Dividing Rational Numbers

In dividing two rational numbers, we must multiply by the reciprocal of the divisor. The RECIPROCAL of a rational number can be found by interchanging the numerator and denominator. For example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, the reciprocal of $\frac{1}{2}$ is $\frac{2}{1}$, the reciprocal of $\frac{-9}{7}$ is $\frac{7}{-9}$, and so on.

> The quotient of two rational numbers can be found by changing the operation to multiplication and giving the reciprocal of the divisor. Then the rule for multiplying two rational numbers can be applied.
>
> i.e. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \overset{\text{MULTIPLY}}{\times} \overset{\text{RECIPROCAL OF } c/d}{\frac{d}{c}} = \frac{a \times d}{b \times c}$

EXAMPLE: $\frac{-5}{12} \div \frac{15}{8} =$ ▨

Solution

Change the operation to multiplication and give the reciprocal of $\frac{15}{8}$.

$$\frac{-5}{12} \div \frac{15}{8} = \frac{-5}{12} \times \overset{\text{RECIPROCAL OF } 15/8}{\frac{8}{15}}$$

Now use the rule for multiplying rational numbers. (Cancel like factors in the numerator and denominator.)

$$= \frac{\overset{-1}{\cancel{-5}} \times \overset{2}{\cancel{8}}}{\underset{3}{\cancel{12}} \times \underset{3}{\cancel{15}}}$$

$$= \frac{-1 \times 2}{3 \times 3}$$

$$= \frac{-2}{9}$$

Find the following quotients. Use cancelling to express your answers in lowest terms.

1. $\frac{5}{8} \div 5 = \frac{5}{8} \times \frac{1}{5} =$

2. $\frac{-14}{9} \div \frac{7}{5} =$

3. $\frac{26}{9} \div \frac{-2}{27} =$

4. $\frac{11}{27} \div \frac{1}{3} =$

III. Adding and Subtracting Rational Numbers

To add (or subtract) two rational numbers with the same denominator, add (or subtract) the numerators. Write the sum (or difference) over the common denominator.

i.e. $\frac{a}{x} + \frac{b}{x} = \frac{a + b}{x}$

$\frac{a}{x} - \frac{b}{x} = \frac{a - b}{x}$

EXAMPLES:

$$\frac{-4}{7} + \frac{2}{7}$$

Add the numerators and put the result over the common denominator, 7.

$$= \frac{-4 + 2}{7}$$

$$= \frac{-2}{7}$$

$$\frac{-8}{13} - \frac{1}{13}$$

Subtract the numerators and put the result over the common denominator, 13.

$$= \frac{-8 - 1}{13}$$

$$= \frac{-9}{13}$$

If two rational numbers do not have the same denominator, they must be expressed with a common denominator before they can be added or subtracted. We change the fractions to their lowest common denominator (smallest number into which both denominators will divide) so that we will be working with the smallest figures possible. Review bottom of page 24 of this lesson where you are shown how to express a rational number with a larger denominator.

EXAMPLE: $\frac{-5}{16} + \frac{-11}{12} =$ ▨

Solution

The smallest number into which both 16 and 12 will divide is 48. (Multiply 16 by 2, 3, 4, etc. until you find a number that 12 will divide into.) In order to express $\frac{-5}{16}$ in 48ths, we must multiply both terms by 3. In order to express $\frac{-11}{12}$ in 48ths, we must multiply both terms by 4.

$$\text{i.e.}\quad \frac{-5}{16} + \frac{-11}{12} = \frac{-5 \times 3}{16 \times 3} + \frac{-11 \times 4}{12 \times 4}$$

$$= \frac{-15}{48} + \frac{-44}{48}$$

Now we can use the rule for adding two rational numbers with like denominators.

$$= \frac{(-15) + (-44)}{48}$$

$$= \frac{-59}{48}$$

When adding or subtracting rational numbers always check your answer to see if it is in lowest terms. If it isn't, divide the numerator and denominator by the appropriate amount in order to put it in lowest terms.

EXAMPLE: $\frac{-5}{9} - \frac{5}{18} =$ ▨

Solution

$$\frac{-5}{9} - \frac{5}{18} = \frac{-5 \times 2}{9 \times 2} - \frac{5}{18}$$

$$= \frac{-10 - 5}{18}$$

$$= \left.\frac{-15}{18}\right\} \text{ Divide numerator and denominator by 3.}$$

$$= \frac{-5}{6} \longleftarrow \text{lowest terms}$$

IV. Complex Fractions

A COMPLEX FRACTION is an expression whose numerator and denominator contain one or more fractions. The following are examples of complex fractions.

$$\frac{\frac{2}{3}}{5}, \qquad \frac{5}{3 + \frac{1}{8}}, \qquad \frac{\frac{1}{2}}{\frac{5}{3}}, \qquad \frac{3 \times \frac{1}{4}}{\frac{1}{6}}$$

When simplifying complex fractions we interpret the long horizontal line to be a sign of division. We first perform any operations that are indicated in the numerator and denominator of the complex fraction. In the last step, we perform the operation of division which is indicated by the long horizontal line.

EXAMPLE: Solution

$$\frac{\frac{2}{3} + \frac{3}{4}}{\frac{1}{2}} = ▨$$

DIVISION SYMBOL REPLACES LONG HORIZONTAL LINE.

$$\frac{\frac{2}{3} + \frac{3}{4}}{\frac{1}{2}} = \left(\frac{2}{3} + \frac{3}{4}\right) \div \frac{1}{2}$$

Perform addition inside brackets first.

$$= \left(\frac{8 + 9}{12}\right) \div \frac{1}{2}$$

$$= \frac{17}{12} \times \frac{2}{1}$$

$$= \frac{17 \times \overset{1}{\cancel{2}}}{\underset{6}{\cancel{12}} \times 1}$$

$$= \frac{17}{6}$$

Simplify the following complex fraction.

$$\frac{1}{-5 \div \frac{2}{3}} = 1 \div (-5 \div \frac{2}{3})$$

$$= 1 \div (-5 \times \frac{}{2})$$

$$= 1 \div \underline{\qquad}$$

$$= 1 \times \underline{\qquad}$$

$$= \underline{\qquad\quad}$$

Self-correcting Exercise #3

Answers may be found on page 59 of this lesson.

1. Give the reciprocal of each rational number.

(a) $\frac{3}{7}$ (b) 5 (c) $\frac{8}{7}$ (d) $\frac{-1}{4}$ (e) $\frac{-2}{5}$

______ ______ ______ ______ ______

2. Express each answer as a single rational number in simplest form.

(a) $\frac{-5}{12} + \frac{-17}{12} =$

(b) $\frac{5}{8} - \frac{-1}{8} =$

(c) $\frac{5}{8} + \frac{5}{6} =$

(d) $\frac{5}{12} - \frac{2}{3} =$

(e) $\frac{-15}{2} \times \frac{4}{5} =$

(f) $35 \div \frac{7}{9} =$

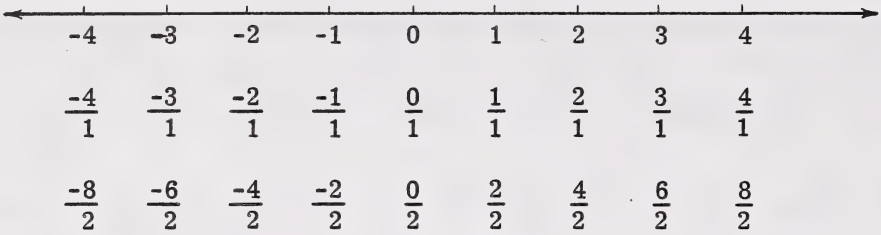
(g) $\frac{-35}{2} \div \frac{-25}{4} =$

(h) $\frac{3}{\frac{1}{4}} =$

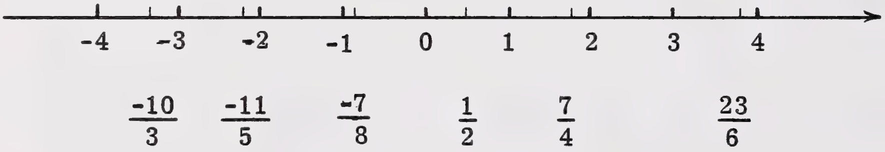
(i) $\frac{\frac{1}{2} - \frac{3}{4}}{4} =$

D. The Rational Number Line

Rational numbers may be represented by points on a number line in the same way that we have represented integers. Some rational numbers are also integers and are coordinates of points already associated with integers. For example, the rational numbers $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$, and so on are all equivalent to the integer 1 and correspond to the same point on the number line as the integer 1.



Other rational numbers lie between the points corresponding to the integers. These rational numbers can be located by dividing the intervals between 0 and 1, 0 and -1, 1 and 2, -1 and -2, and so on, into halves, thirds, quarters, fifths, etc. If this process of division were carried on indefinitely, every rational number could be located on the number line. Note that a few of these rational numbers have been located on the number line below.



One rational number is greater than another if it lies to the right of the other on the number line. For example, from the number line above we know that $\frac{7}{4} > \frac{1}{2}$ since $\frac{7}{4}$ lies to the right of $\frac{1}{2}$ on the number line.

Fill in the blanks below.

1. $\frac{-11}{5}$ ______ $\frac{-7}{8}$ since $\frac{-11}{5}$ lies to the ____________ of $\frac{-7}{8}$

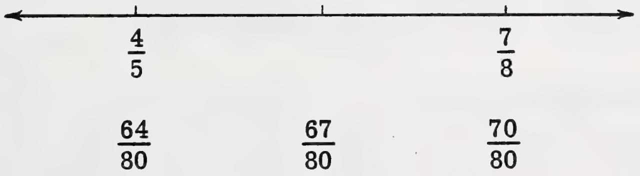
2. $\frac{7}{4}$ ______ 1 since $\frac{7}{4}$ lies to the ____________ of 1.

3. $\frac{23}{6}$ ______ 4 since $\frac{23}{6}$ lies to the ____________ of 4.

The rational numbers are said to be DENSE because between any two rational numbers we can locate another rational number. For example, consider the rational numbers $\frac{4}{5}$ and $\frac{7}{8}$. To find a rational number that is between them, we can find their average (arithmetic mean).

$$\frac{\frac{4}{5}+\frac{7}{8}}{2} = \frac{\frac{32}{40}+\frac{35}{40}}{2} = \frac{67}{40} \div \frac{2}{1} = \frac{67}{40} \times \frac{1}{2} = \frac{67}{80}$$

The point corresponding to $\frac{67}{80}$ lies midway between the points on the number line whose coordinates are $\frac{4}{5}$ and $\frac{7}{8}$.



Even though the rational numbers are dense, we cannot conclude that they fill up a solid number line. In fact, the rational number line must be regarded as being dense but porous. There are still some gaps in the line which can be filled up with points which represent numbers that do not belong to set Q. In Topic 3 of this lesson, we will locate some new numbers on the line that will fill up these gaps. Since the rational numbers are not represented by all the points on a solid line, we say that the rational numbers are NOT COMPLETE. That is, set Q has the density property but not the completeness property.

E. Properties of the Rational Numbers

The set of rational numbers satisfies the same properties as the integers under the operations of addition and multiplication. (Review chart on page 14 of this lesson.) But, the rational numbers have a new property under the operation of multiplication that does not apply to the set of integers. Every rational number (except zero) has an inverse under the operation of multiplication. (Zero has no inverse since $\frac{1}{0}$ is undefined.)

The product of a rational number and its multiplicative inverse is one.

DEFINITION-MULTIPLICATIVE INVERSE

For every rational number $\frac{a}{b}$ (where $a \neq 0$), there is a unique rational number $\frac{b}{a}$ such that:

$$\frac{a}{b} \times \frac{b}{a} = 1$$

The rational numbers $\frac{a}{b}$ and $\frac{b}{a}$ are multiplicative inverses of each other.

EXAMPLES:

1. $\frac{3}{5}$ and $\frac{5}{3}$ are multiplicative inverses since $\frac{3}{5} \times \frac{5}{3} = 1$.
2. $\frac{-1}{2}$ and -2 are multiplicative inverses since $\frac{-1}{2} \times -2 = 1$.

Multiplicative inverses do not exist in the set of whole numbers or the set of integers.

Below is a summary of the properties of set Q under the operations of addition and multiplication. Fill in the blanks.

For any rational numbers $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$:	Addition	Multiplication
Closure Properties	$\left(\frac{a}{b}+\frac{c}{d}\right)$ is a rational number.	$\left(\frac{a}{b}\times\frac{c}{d}\right)$ is a ________ number.
________ Properties	$\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\underline{\quad}$	$\frac{a}{b}\times\frac{c}{d}=\underline{\quad}\times\frac{a}{b}$
________ Properties	$\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\underline{\quad}+\underline{\quad}\right)+\frac{e}{f}$	$\frac{a}{b}\left(\frac{c}{d}\times\frac{e}{f}\right)=\left(\underline{\quad}\times\frac{c}{d}\right)\underline{\quad}$
________ Property	$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)=\left(\frac{a}{b}\times\underline{\quad}\right)+\left(\underline{\quad}\times\underline{\quad}\right)$	
________ Elements	Additive identity is 0. $\frac{a}{b}+0=\underline{\quad}$	Mult. identity is ____. $\frac{a}{b}\times\underline{\quad}=\underline{\quad}$
Inverse Elements	Additive inverse of $\frac{a}{b}$ is $\frac{-a}{b}$. $\frac{a}{b}+\frac{-a}{b}=\underline{\quad}$	* Mult. inverse of $\frac{a}{b}$ is ____. $\frac{a}{b}\times\underline{\quad}=\underline{\quad}$

* This is a property of the rational numbers but not of the integers or whole numbers.

A set of numbers which has all the above properties with respect to the operations of addition and multiplication is called a FIELD. Sets W and I lack some of the properties of a field.

Like set I, set Q is closed under the operations of addition, multiplication, and subtraction. But, it has the added feature that it is also closed under the operation of division (except for division by zero). This means that the quotient of two rational numbers is always a rational number (except when the divisor is zero).

EXAMPLES:

$$\left.\begin{array}{ll} 1. & \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{1 \times \overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}} \times 3} = \frac{1 \times 2}{1 \times 3} = \boxed{\frac{2}{3}} \\[2ex] 2. & \frac{-2}{5} \div 5 = \frac{-2}{5} \times \frac{1}{5} = \frac{-2 \times 1}{5 \times 5} = \boxed{\frac{-2}{25}} \end{array}\right\} \text{Results are rational numbers.}$$

EXERCISE - The Set of Rational Numbers

1. Fill in the blanks.

(a) An equal fraction is obtained if both terms of the fraction are

__________ or __________ by the same non-zero

number.

(b) Since every integer is a rational number, the set of integers is a

__________ of the set of rational numbers.

(c) One rational number is greater than another if it lies to the

__________ of the other on the number line.

(d) Rational numbers are said to be __________ because

between any two rational numbers, another rational number can be

located.

(e) The rational numbers are not __________ because they do not

represent all the points on a solid line.

(f) The smallest denominator that two fractions can be changed to is

called the __________ __________ denominator of

the two fractions.

(g) $\frac{5}{4}$ and $\frac{4}{5}$ are ________________ inverses because their ________________ is one.

(h) The quotient of $\frac{2}{3}$ and $\frac{3}{5}$ is determined by changing the operation to ______________________ and giving the ________________________ of __________.

(i) $\frac{2}{3}$ is a __________________ rational number while $\frac{-2}{3}$ is a ________________ rational number. These numbers are ________________ inverses of each other.

(j) The operation of division is closed in set Q except for division by ________________.

2. Express each answer as a single rational number in lowest terms.

(a) $\frac{-3}{28} + \frac{17}{28} =$

(b) $\frac{7}{8} - \frac{5}{8} =$

(c) $\frac{-1}{4} + \frac{-3}{4} =$

(d) $\frac{3}{25} + \frac{9}{75} =$

(e) $\frac{7}{24} - \frac{5}{18} =$

(f) $\frac{5}{8} - \frac{11}{12} =$

(g) $\frac{-27}{9} \times \frac{14}{35} =$

(h) $\frac{25}{8} \times \frac{24}{5} =$

(i) $-12 \times \frac{-3}{8} =$

(j) $\frac{3}{8} \div 4 =$

(k) $\frac{-2}{3} \div \frac{4}{9} =$

(l) $\frac{4}{15} \div \frac{14}{45} =$

(m) $3 \times \frac{7}{6} \times \frac{4}{21} =$

(n) $\frac{2}{3} + \frac{5}{6} + \frac{7}{12} =$

(o) $\dfrac{5}{\frac{1}{2} + \frac{1}{3}} =$

(p) $\dfrac{3 \times \frac{1}{4}}{\frac{1}{6}} =$

(q) $\dfrac{\frac{3}{4}}{\frac{1}{2} - \frac{5}{2}} =$

(r) $\dfrac{-2}{\frac{1}{5}} =$

3. Fill in the blanks to make each statement true.

(a) $\frac{3}{15} = \frac{\quad}{30}$ (b) $\frac{\quad}{13} = \frac{-4}{26}$ (c) $\frac{-2}{9} = \frac{\quad}{72}$

(d) $\frac{5}{3} = \frac{105}{\quad}$ (e) $\frac{4}{8} = \frac{6}{\quad}$ (f) $\frac{\quad}{35} = \frac{9}{105}$

4. If x can be replaced by any value from A where A = $\{-1, \frac{-1}{4}, 0, \frac{1}{4}, 1\}$, state the values of x that make each of the following statements true.

(a) $x > \frac{-1}{4}$ (b) $x \leqq \frac{1}{4}$ (c) $x \geqq -1$

$x = 0, \frac{1}{4}, 1$ ________ ________

(d) $x > 1$ (e) $\frac{-1}{4} \leqq x \leqq \frac{1}{4}$ (f) $0 < x < 1$

________ ________ ________

5. Give one example to show that subtraction is not associative in set Q.

Left Side	Right Side
$\left(\frac{5}{8} - \frac{3}{8}\right) - \frac{1}{8}$	$\frac{5}{8} - \left(\frac{3}{8} - \frac{1}{8}\right)$
$= \_\_\_ - \frac{1}{8}$	$= \frac{5}{8} - \_\_\_$
$= \_\_\_$	$= \_\_\_$

Are the two sides equal? ______

6. Give one example to show that subtraction is not commutative in set Q.

Left Side	Right Side
$\frac{3}{4} - \frac{5}{8}$	$\frac{5}{8} - \frac{3}{4}$
=	=

Are the two sides equal? ______

7. Give one example to show that division is not associative in set Q.

Left Side	Right Side
$\left(\frac{2}{3} \div \frac{3}{5}\right) \div \frac{5}{8}$	
=	

Are the two sides equal? ______

8. Give one example to show that division is not commutative in set Q.

Left Side	Right Side
$\frac{5}{8} \div \frac{25}{16}$	
=	

Are the two sides equal? ______

9. Give the additive and multiplicative inverses of each of the following rational numbers.

	Additive Inverse	Multiplicative Inverse
(a) $\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{3}$
(b) $\frac{-1}{2}$		
(c) 3		
(d) $\frac{8}{5}$		
(e) 0		

(See bottom of p. 20)

Topic Three: The Set of Real Numbers

A. Relationship Between Rational Numbers and Decimal Numbers

In order to find a decimal name for a rational number of the form $\frac{a}{b}$, we must divide "b" into "a".

EXAMPLE 1:

$\frac{23}{4} = \underline{5.75}$

$$\begin{array}{r} 5.75 \\ 4\overline{)23.\underset{\times\times}{00}} \\ \underline{20} \\ 3\;0 \\ \underline{2\;8} \\ 20 \\ \underline{20} \end{array}$$

EXAMPLE 2:

$\frac{27}{200} = \underline{0.135}$

$$\begin{array}{r} .135 \\ 200\overline{)27.0\underset{\times\times}{00}} \\ \underline{20\;0} \\ 7\;00 \\ \underline{6\;00} \\ 1\;000 \\ \underline{1\;000} \end{array}$$

Decimals like 5.75 and 0.135 are called TERMINATING DECIMALS. They are obtained by dividing two integers and at some point in the division process, a remainder of zero is reached. However, not all rational numbers can be represented by terminating decimals. Sometimes when we divide one integer into another, we will never obtain a remainder of zero no matter how far we carry the division process. Instead, the remainders start repeating and one or more repeating digits appear in the quotient.

EXAMPLE 1:

$\frac{2}{3} = 0.\overline{6}$

We place a bar above the digit 6 to indicate that it repeats indefinitely.

$$\begin{array}{r} .666\ldots \\ 3\overline{)2.0\underset{\times\times}{00}} \\ \underline{1\;8} \\ \textcircled{2}0 \\ \underline{18} \\ 2 \end{array}$$

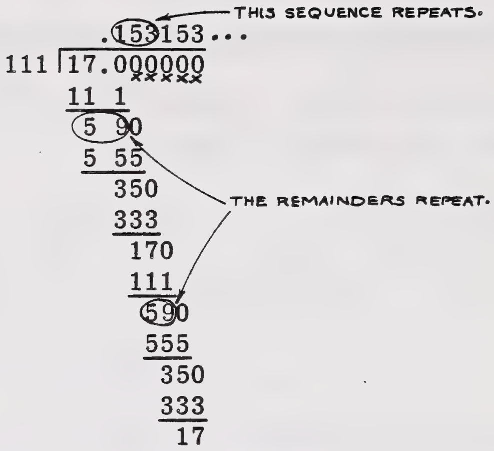
DIGIT 6 IN THE QUOTIENT KEEPS REPEATING.

REMAINDER 2 KEEPS OCCURRING.

EXAMPLE 2:

$$\frac{17}{111} = 0.\overline{153}$$

The bar above the digits 153 indicates that this entire sequence repeats.


THIS SEQUENCE REPEATS.
.153153...
111 | 17.000000
11 1
5 90
5 55
350
333
170
111
590
555
350
333
17
THE REMAINDERS REPEAT.

Decimals like $0.\overline{6}$ and $0.\overline{153}$ are called NON-TERMINATING REPEATING DECIMALS. They are obtained by dividing two integers and at some point in the division process, the remainders start repeating.

When the denominator of a rational number is divided into the numerator, a remainder occurs at each step. If at one stage of the division the remainder is zero, the division is complete and a terminating decimal is obtained. If a remainder of zero is not obtained, some remainder will occur for a second time, and a non-terminating repeating decimal is obtained.

Every rational number can be represented by either a terminating decimal or a non-terminating repeating decimal.

Self-correcting Exercise #4

Answers may be found on page 60 of this lesson.

1. Express each of the following rational numbers as a decimal. Then state whether the decimal is "terminating" or "non-terminating repeating". (Room for rough work is provided below.)

(a) $\frac{4}{11}$ = __________ Classification: ______________________

(b) $\frac{17}{8}$ = __________ Classification: ______________________

(c) $\frac{15}{7}$ = __________ Classification: ______________________

(d) $\frac{17}{20}$ = __________ Classification: ______________________

Rough Work

You were just shown how to write a rational number in decimal form. Now consider the reverse problem - how can you express a decimal in the rational form $\frac{a}{b}$?

In the case of a terminating decimal, it is easy to write it in rational form. You merely write the digits that appear in the decimal over the value of the place held by the *last* digit in the decimal.

EXAMPLES:

$$0.319 = \frac{319 \leftarrow \text{THESE DIGITS OCCUR IN THE DECIMAL.}}{1000 \leftarrow \text{THE LAST DIGIT, 9, IS IN THE THOUSANDTHS PLACE.}}$$

$$0.0007 = \frac{7}{10,000 \leftarrow \text{THE LAST DIGIT, 7, IS IN THE TEN-THOUSANDTHS PLACE.}}$$

Any non-terminating repeating decimal may also be written in rational form. Study the following examples.

EXAMPLE 1: Write the decimal $0.\overline{4}$ in rational form.

Solution

Let x represent the decimal $0.\overline{4}$.

$$x = 0.\overline{4} = 0.4444\ldots$$

$$\begin{array}{rcl} 10x & = & 4.444\ldots \\ x & = & 0.444\ldots \\ \hline 9x & = & 4 \\ & & \\ x & = & \dfrac{4}{9} \end{array} \qquad \text{(subtract)}$$

(Multiply by 10 so that the decimal occurs after the first sequence of recurring digits.)

The non-terminating repeating decimal $0.\overline{4}$ is equivalent to the rational number $\frac{4}{9}$.

EXAMPLE 2: Write the decimal $0.\overline{56}$ in rational form.

Solution

Let x represent the decimal $0.\overline{56}$.

$$x = 0.\overline{56} = 0.5656\ldots$$

$$\begin{array}{rcl} 100x & = & 56.56\ldots \\ x & = & 0.56\ldots \\ \hline 99x & = & 56 \\ & & \\ x & = & \dfrac{56}{99} \end{array}$$

(Multiply by 100 so that the decimal occurs after the first sequence of recurring digits, 56.)

The non-terminating repeating decimal $0.\overline{56}$ is equivalent to the rational number $\frac{56}{99}$.

Every terminating or non-terminating repeating decimal can be expressed as a rational number.

We can conclude that there is a one-to-one correspondence between the set of rational numbers and the set of terminating and non-terminating repeating decimals. This means that:

1. Every rational number can be associated with exactly one decimal that either terminates or repeats.

2. Every decimal that terminates or repeats can be associated with exactly one rational number.

Self-correcting Exercise #5

Answers may be found on page 61 of this lesson.

1. Express each decimal as a rational number in lowest terms.

(a) 0.125

(b) 0.0004

(c) $0.\overline{153}$

(d) $0.\overline{7}$

B. Irrational Numbers

On page 33 of this lesson, you were told that the rational number line is porous. This means that there are gaps in the line that are not filled up with rational numbers. In this section, we will discuss a new set of numbers which fills up the gaps in the rational number line.

There is a one-to-one correspondence between the set of rational numbers and the set of terminating and non-terminating repeating decimals. But, there are an infinite number of decimals that neither terminate nor repeat. Decimals like 3.010010001..., 13.2764..., 0.314798... are of this type. The set of numbers which can be represented by non-terminating, non-repeating decimals is called the SET OF IRRATIONAL NUMBERS and is represented by the capital letter T.

$$\text{i.e.}\quad T = \{\text{all irrational numbers}\}$$
$$= \{\text{all non-terminating, non-repeating decimals}\}$$

Irrational numbers are not always written in decimal form. For example, the symbol $\sqrt{2}$ represents the positive irrational number whose square is 2. $\sqrt{2}$ represents the same irrational number as the non-terminating, non-repeating decimal 1.4142136 ... . The symbol $-\sqrt{2}$ represents the negative irrational number whose square is 2. $-\sqrt{2}$ represents the same irrational number as the non-terminating, non-repeating decimal -1.4142136 ... .

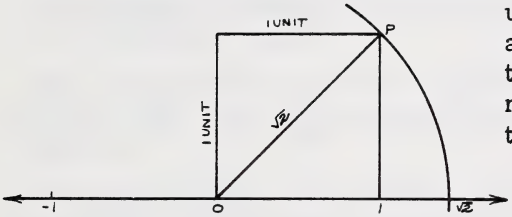
The symbol π represents the same irrational number as the non-terminating, non-repeating decimal 3.1415926 ... . For all circles, π is the ratio of the circumference of the circle to the diameter.

If should be understood that an irrational number is an exact number. For example, the irrational number $\sqrt{2}$ is an exact number and often appears in mathematics in this form. However, when an irrational number is required in decimal form for computation purposes, the value is approximated by using a rational number. For example, the value of the irrational number $\sqrt{2}$ is often approximated by the rational number 1.414. The number 1.414 is said to be a RATIONAL APPROXIMATION of $\sqrt{2}$. Similarly, the rational approximations $\frac{22}{7}$ and 3.14 are often used for the irrational number π.

On page 57 of this lesson, you will find a table of square roots for integers between 1 and 100 inclusive. In this table, some values of $\sqrt{n}$ are exact while others are only rational approximations. For example, $\sqrt{25}$ is exactly equal to 5, while 5.196 is only a rational approximation of $\sqrt{27}$.

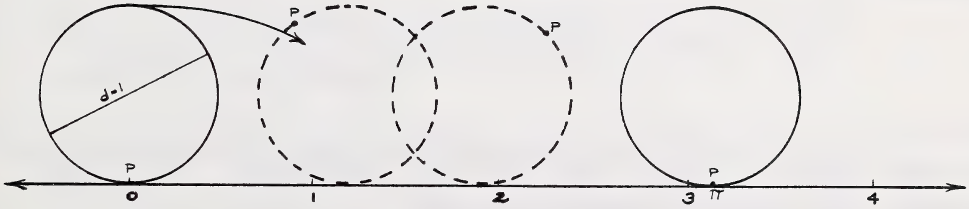
Irrational numbers are coordinates of definite points on the number line.

For example, the irrational number $\sqrt{2}$ can be located on the number line by drawing a square that is 1 unit on a side. According to the Theorem of Pythagoras, the diagonal of this square must be $\sqrt{1^2 + 1^2} = \sqrt{2}$ units. If a compass is used to measure off the length of this diagonal, this measurement can be copied onto the number line.



The square has a diagonal of $\sqrt{2}$ units. An arc can be drawn with centre at the origin and radius the length of the diagonal. This arc cuts the number line at a point $\sqrt{2}$ units from the origin.

The irrational number π can also be located on the number line. A circle with a diameter of 1 unit has a circumference of π units since $C = \pi d = \pi \times 1 = \pi$ units. If such a circle is rolled along a number line for one complete revolution starting with point P on the circumference coinciding with the origin on the number line, a point corresponding to π can be determined.



In general, any irrational number corresponds to one and only one point on the number line.

Self-correcting Exercise #6

Answers may be found on page 61 of this lesson.

1. Classify each of the following numbers as being rational or irrational.

(a) $2.\overline{45}$ ____________ (b) $0.2123478\ldots$ ____________ (c) $\frac{5}{9}$ ____________

(d) $0.3333\ldots$ ____________ (e) 13.675 ____________ (f) $0.325325532555\ldots$ ____________

(g) $\sqrt{3}$ ____________ (h) $\sqrt{16}$ ____________ (i) 0 ____________

2. Use the table on page 57 to find the following square roots.

(a) $\sqrt{35}$ = _____ (b) $\sqrt{84}$ = _____ (c) $\sqrt{76}$ = _____ (d) $\sqrt{8}$ = _____

C. The Set of Real Numbers and Its Properties

We have discussed the set of rational numbers Q which may be represented by terminating or non-terminating repeating decimals and the set of irrational numbers T which may be represented by non-terminating, non-repeating decimals.

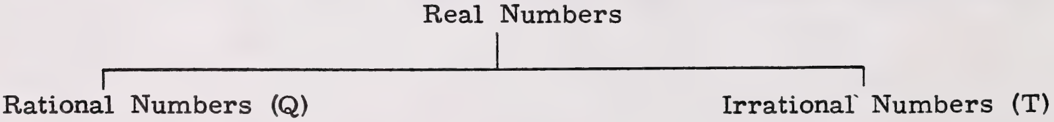
Set Q and set T are disjoint sets because they have no elements in common.

$$\text{i.e.}\quad Q \cap T = \{\text{all rationals}\} \cap \{\text{all irrationals}\}$$
$$= \{\ \}$$

If we take the union of set Q and set T, we will arrive at a new set of elements which we call the SET OF REAL NUMBERS. We always designate this set by the capital letter R.

$$\text{i.e.}\quad Q \cup T = \{\text{all rationals}\} \cup \{\text{all irrationals}\}$$
$$= \{\text{all real numbers}\}$$
$$= R$$

The following diagram illustrates the composition of set R.



There is a one-to-one correspondence between the set of real numbers and the set of points in a solid number line. This means that:

1. Every real number can be represented by a point in the number line.

2. Every point in the number line can be associated with a real number.

Since the real numbers "cover" the number line, set R has the COMPLETENESS PROPERTY. (On page 33 of this lesson, you were told that the rational numbers are not complete because they do not fill up the number line.)

Like set Q, set R has the DENSITY PROPERTY. This means that between any two real numbers another real number can always be located.

The real numbers are ordered in the same way as the other sets of numbers. If the point corresponding to the real number "a" lies to the right of the point corresponding to the real number "b", then $a > b$. For example $3.45... > 3.42...$ since the point corresponding to 3.45... lies to the right of the point corresponding to 3.42... .

The set of real numbers together with the operations of addition and multiplication and some of their properties form the REAL NUMBER SYSTEM. Since the real number system satisfies all the properties that we have already attributed to the rational numbers under the operations of addition and multiplication, it must also form a FIELD. List the eleven properties of a field as they apply to set R. (Review chart on page 35 of this lesson.)

1. Set **R** is closed under ______________________________.

2. Set R is closed under ______________________________.

3. Addition is commutative.

4. ______________________ is ______________________.

5. ______________________ is ______________________.

6. Multiplication is associative.

7. ______________________ distributes over ______________________.

8. _______________ identity is __________.

9. _______________ identity is __________.

10. ____________________ inverse of "a" is ____________.

11. ____________________ inverse of "a" is ____________.

D. Relationship Between Number Sets

The sets of numbers which we have discussed so far are related to each other in the following manner.

1. Sets N, W, I, and Q are all proper subsets of set R. This means that every natural number, whole number, integer, and rational number is also a real number. For example,

   205 is a natural number, but it is also a real number.
   0 is a whole number, but it is also a real number.
   -7 is an integer, but it is also a real number.
   $\frac{-2}{3}$ is a rational number, but it is also a real number.

   Set R includes the irrational numbers which do not belong to sets N, W, I or Q. For example,

   $\sqrt{2}$ is an irrational number, but it is not a natural number, whole number, integer, or rational number.

2. Sets N, W, and I are all proper subsets of set Q. This means that every natural number, whole number, and integer is also a rational number.
   For example,

   205 is a natural number, but it is also a rational number.
   0 is a whole number, but it is also a rational number.
   -7 is an integer, but it is also a rational number.

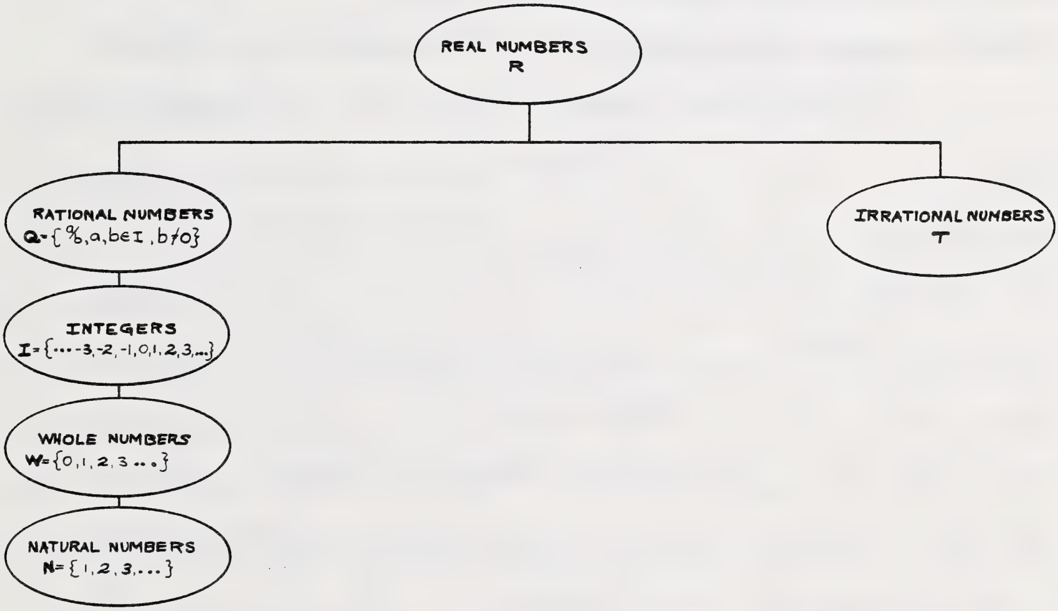
   Set Q includes fractions of the form $\frac{a}{b}$, where $|b|$ is not a factor of $|a|$, that do not belong to sets N, W, or I.
   For example,

   $\frac{-2}{3}$ is a rational number, but it is not a natural number, whole number or integer.

3. Sets N and W are both proper subsets of set I. This means that every natural number and every whole number is also an integer.
   For example,
   205 is a natural number, but it is also an integer.
   0 is a whole number, but it is also an integer.
   Set I includes negative integers that do not belong to set N or set W.
   For example,
   -7 is an integer, but it is not a natural number or a whole number.

4. Set N is a proper subset of set W. This means that every natural number is also a whole number.
   For example,
   205 is a natural number, but it is also a whole number.
   Set W contains the one element zero which does not belong to set N.
   That is,
   0 is a whole number, but it is not a natural number.

The following diagram shows the relationship between set R and its various subsets.


REAL NUMBERS
R
RATIONAL NUMBERS
Q={a/b, a,b∈I, b≠0}
IRRATIONAL NUMBERS
T
INTEGERS
I={...-3,-2,-1,0,1,2,3,...}
WHOLE NUMBERS
W={0,1,2,3...}
NATURAL NUMBERS
N={1,2,3,...}

Use the above diagram to answer the following questions.

Yes or No?

1. Is set I a subset of set Q? ____________
2. Is set W a subset of set N? ____________
3. Is set T a subset of set R? ____________
4. Are set Q and set I disjoint sets? ____________
5. Are set Q and set T disjoint sets? ____________
6. Are set I and set T disjoint sets? ____________

Most of the work in this course is limited to real numbers. Do not make the mistake, however, of thinking that you are finished with the development of the number system. In the ninth lesson of this course you will be introduced to numbers like $\sqrt{-7}$ and $2\sqrt{-5}$ that are not real numbers. They belong to a new set of numbers called the set of imaginary numbers. The set of real number together with the set of imaginary numbers make up the set of complex numbers.

Self-correcting Exercise #7

Answers may be found on page 62 of this lesson.

1. Decide whether each of the following statements is true or false?

True or False?

(a) $5 \in I$ ____________

(b) $0 \in N$ ____________

(c) $-2 \in Q$ ____________

(d) $0.\overline{3} \in R$ ____________

(e) $\pi \in Q$ ____________

(f) $-\sqrt{9} \in I$ ____________

(g) $\frac{4}{5} \in I$ ____________

(h) $\sqrt{10} \in Q$ ____________

(i) $0.2 \in Q$ ____________

(j) $3.145... \in R$ ____________

(k) $1.\overline{56} \in T$ ____________

(l) $\sqrt{-2} \in R$ ____________

EXERCISE - The Set of Real Numbers

1. Fill in the blanks.

(a) The real numbers are said to be ____________________ because they are in one-to-one correspondence with the set of points in a line.

(b) Since $Q \cap T = \{\ \}$, set Q and T are ________________ sets. The union of sets Q and T is the set of ________________ numbers.

(c) We know that $3 \times \sqrt{5} = \sqrt{5} \times 3$ because multiplication is ____________________ in set R.

(d) $1 \times \sqrt{7} = \sqrt{7}$ since one is the __________________ ____________ in set R.

(e) $(2 + \sqrt{3}) \in R$ since the set of ______________ numbers is ____________________ under addition.

(f) The set of integers is a subset of the set of ________________ numbers and the set of _________________ numbers.

(g) All ____________________ numbers can be represented by non-terminating, non-repeating decimals.

(h) The number 0.45 is called a __________________ decimal while the number $0.\overline{45}$ is called a __________________________ decimal.

(i) The ___________ inverse of $\sqrt{2}$ is $-\sqrt{2}$.

(j) The number $\frac{22}{7}$ is a ____________ approximation of the _____________ number π.

2. Express each of the following rational numbers as a decimal. (Space for rough work is provided below.)

   (a) $\frac{5}{3}$ = ______ (b) $\frac{3}{500}$ = ______ (c) $\frac{32}{15}$ = ______ (d) $\frac{13}{8}$ = ______

Rough Work

3. Express each terminating decimal as a rational number in lowest terms.

   (a) 0.35 (b) 0.0008

4. Express each repeating decimal as a rational number in lowest terms.

   (a) $0.\overline{24}$ (b) $4.\overline{5}$

5. Find decimal approximations for each of the following expressions using the table of square roots on page 57.

   (a) $5 + 3\sqrt{2}$
   $= 5 + 3\ (1.414)$
   $=$

   (b) $\frac{8 - \sqrt{39}}{3}$
   $=$

(c) $\frac{3}{2}\sqrt{47}$

=

(d) $\frac{\sqrt{5} + \sqrt{13}}{2}$

=

6. Put check marks in the appropriate columns to indicate which sets each number belongs to.

	N	W	I	Q	R
(a) -5			✓	✓	✓
(b) 2					
(c) $\frac{-5}{9}$					
(d) $0.3654...$					
(e) $\sqrt{16}$					
(f) $\sqrt{2}$					
(g) π					
(h) -10.1					
(i) 0					
(j) $0.\overline{15}$					
(k) $\sqrt{-5}$					
(l) $\frac{2}{3}$					

7. Decide whether each of the following statements is true or false.

	True or False?
(a) The multiplicative inverse of $\sqrt{2}$ is $\frac{1}{\sqrt{2}}$.	true
(b) 0 is the additive identity in set R.	
(c) $I \subseteq N$	
(d) 0.1010010001... is a repeating decimal.	
(e) $3.\overline{165}$ is a rational number.	
(f) All irrational numbers are real numbers.	
(g) The distributive property holds in set R.	
(h) The elements of set Q have no additive inverses.	
(i) The product of two real numbers is always a real number.	
(j) All decimals belong to set Q.	
(k) Some elements of Q are also integers.	
(l) Some numbers belong to both sets Q and T.	
(m) Multiplicative inverses exist in sets Q and R but not in sets N, W, and I.	
(n) Set R is closed under addition, subtraction, multiplication, and division (except for division by zero).	
(o) Set W is closed under addition and subtraction.	
(p) Subtraction is commutative in set I.	
(q) The additive inverse of π is $-\pi$.	

TABLE OF SQUARE ROOTS

n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$
1	1.000	26	5.099	51	7.141	76	8.718
2	1.414	27	5.196	52	7.211	77	8.775
3	1.732	28	5.292	53	7.280	78	8.832
4	2.000	29	5.385	54	7.348	79	8.888
5	2.236	30	5.477	55	7.416	80	8.944
6	2.449	31	5.568	56	7.483	81	9.000
7	2.646	32	5.657	57	7.550	82	9.055
8	2.828	33	5.745	58	7.616	83	9.110
9	3.000	34	5.831	59	7.681	84	9.165
10	3.162	35	5.916	60	7.746	85	9.220
11	3.317	36	6.000	61	7.810	86	9.274
12	3.464	37	6.083	62	7.874	87	9.327
13	3.606	38	6.164	63	7.937	88	9.381
14	3.742	39	6.245	64	8.000	89	9.434
15	3.873	40	6.325	65	8.062	90	9.487
16	4.000	41	6.403	66	8.124	91	9.539
17	4.123	42	6.481	67	8.185	92	9.592
18	4.243	43	6.557	68	8.246	93	9.644
19	4.359	44	6.633	69	8.307	94	9.695
20	4.472	45	6.708	70	8.367	95	9.747
21	4.583	46	6.782	71	8.426	96	9.798
22	4.690	47	6.856	72	8.485	97	9.849
23	4.796	48	6.928	73	8.544	98	9.899
24	4.899	49	7.000	74	8.602	99	9.950
25	5.000	50	7.071	75	8.660	100	10.000

Key to Self-correcting Exercises in Lesson B

Exercise #1, page 4

1. (a) -4 lies to the left of -1; $-4 < -1$
   (b) 0 lies to the right of -9; $0 > -9$
   (c) -6 lies to the right of -8; $-6 > -8$
   (d) 12 lies to the right of 5; $12 > 5$
   (e) -3 lies to the left of 1; $-3 < 1$

2. (a) -10 (b) 30 (c) -5 (d) -50
   (e) 20 (f) 0 (g) 100 (h) -15

3. (a) $|-11| = 11$ (b) $|0| = 0$ (c) $|8| = 8$ (d) $|-1| = 1$

4. (a) -14 (b) 50 (c) -1 (d) 15

Exercise #2, page 11

1. (a) $6 - 9 = 6 + (-9) = \underline{-3}$ (b) $17 - (-8) = 17 + 8 = \underline{25}$
   (c) $-13 - (-4) = -13 + 4 = \underline{-9}$ (d) $-7 - 5 = -7 + (-5) = \underline{-12}$
   (e) $-2 - (-2) = -2 + 2 = \underline{0}$ (f) $3 - 6 = 3 + (-6) = \underline{-3}$

2.

	Are the signs alike or different?	Will the result be positive or negative?	Result
(a) $16 \div -4$	different	negative	-4
(b) $4 + (-7)$	different	negative	-3
(c) 6×-5	different	negative	-30
(d) $\frac{-2}{-1}$	alike	positive	2
(e) $(-3) + (-1)$	alike	negative	-4
(f) $\frac{14}{-2}$	different	negative	-7
(g) -3×-3	alike	positive	9
(h) $-9 + 13$	different	positive	4
(i) $-10 \div -5$	alike	positive	2

Exercise #2, continued

3. (a) $6 \times -5 = \underline{-30}$

(b) $-8 \div -1 = \underline{8}$

(c) $-15 + 12 = \underline{-3}$

(d) $5 - 12 = 5 + (-12) = \underline{-7}$

(e) $-4 \times -4 = \underline{16}$

(f) $\frac{-21}{7} = \underline{-3}$

(g) $-3 - (-15) = -3 + 15 = \underline{12}$

(h) $17 - (-1) = 17 + 1 = \underline{18}$

(i) $(-6) + (-6) = \underline{-12}$

(j) $-6 + 6 = \underline{0}$

(k) $-3 - 2 = -3 + (-2) = \underline{-5}$

(l) $\frac{10}{-1} = \underline{-10}$

(m) $-8 + 9 = \underline{1}$

(n) $-6 - (-4) = -6 + 4 = \underline{-2}$

(o) $-5 + 2 - 8 + 4 = -13 + 6 = \underline{-7}$

(p) $3 - 9 - 1 + 12 = 15 - 10 = \underline{5}$

(q) $-6 + 10 + 3 - 9 = -15 + 13 = \underline{-2}$

(r) $2 - (-4) - 8 + 7 = 2 + 4 - 8 + 7 = 13 - 8 = \underline{5}$

(s) $\frac{7 - 12}{-5} = \frac{7 + (-12)}{-5} = \frac{-5}{-5} = \underline{1}$

(t) $\frac{-8 - 12}{-4} = \frac{-8 + (-12)}{-4} = \frac{-20}{-4} = \underline{5}$

(u) $\frac{-4(2 - 6)}{-2} = \frac{-4[2 + (-6)]}{-2} = \frac{-4(-4)}{-2} = \frac{16}{-2} = \underline{-8}$

(v) $-1(8 - 4 - 3 - 6) = -1(8 - 13) = -1[8 + (-13)] = -1(-5) = \underline{5}$

Exercise #3, page 31

1. (a) $\frac{7}{3}$ (b) $\frac{1}{5}$ (c) $\frac{7}{8}$ (d) -4 (e) $\frac{-5}{2}$

2. (a) $\frac{-5}{12} + \frac{-17}{12} = \frac{(-5) + (-17)}{12} = \frac{-22}{12} = \enclose{circle}{\frac{-11}{6}}$

(b) $\frac{5}{8} - \frac{-1}{8} = \frac{5 - (-1)}{8} = \frac{5 + 1}{8} = \frac{6}{8} = \enclose{circle}{\frac{3}{4}}$

(c) $\frac{5}{8} + \frac{5}{6} = \frac{5 \times 3}{8 \times 3} + \frac{5 \times 4}{6 \times 4} = \frac{15 + 20}{24} = \enclose{circle}{\frac{35}{24}}$

(d) $\frac{5}{12} - \frac{2}{3} = \frac{5}{12} - \frac{2 \times 4}{3 \times 4} = \frac{5 - 8}{12} = \frac{-3}{12} = \enclose{circle}{\frac{-1}{4}}$

Exercise #3, continued

2. (e) $\frac{-15}{2}\times\frac{4}{5} = \frac{\overset{-3}{\cancel{-15}}\times\overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}}\times\underset{1}{\cancel{5}}} = \frac{-3\times 2}{1\times 1} = \boxed{-6}$

(f) $35 \div \frac{7}{9} = \frac{35}{1}\times\frac{9}{7} = \frac{\overset{5}{\cancel{35}}\times 9}{1\times\underset{1}{\cancel{7}}} = \boxed{45}$

(g) $\frac{-35}{2}\div\frac{-25}{4} = \frac{-35}{2}\times\frac{4}{-25} = \frac{\overset{-7}{\cancel{-35}}\times\overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}}\times\underset{-5}{\cancel{-25}}} = \frac{-7\times 2}{1\times -5} = \frac{-14}{-5} = \boxed{\frac{14}{5}}$

(h) $\frac{3}{\frac{1}{4}} = 3\div\frac{1}{4} = \frac{3}{1}\times\frac{4}{1} = \boxed{12}$

(i) $\frac{\frac{1}{2}-\frac{3}{4}}{4} = \left(\frac{1}{2}-\frac{3}{4}\right)\div 4 = \left(\frac{2-3}{4}\right)\div 4 = \frac{-1}{4}\times\frac{1}{4} = \boxed{\frac{-1}{16}}$

Exercise #4, page 43

1. (a) $\frac{4}{11} = 0.\overline{36}$ (non-terminating repeating)

(b) $\frac{17}{8} = 2.125$ (terminating)

(c) $\frac{15}{7} = 2.\overline{142857}$ (non-terminating repeating)

(d) $\frac{17}{20} = 0.85$ (terminating)

$$\begin{array}{r}
.3636\ldots \\
11\,\overline{)4.0000} \\
\underline{3\,3}^{\times\times\times} \\
70 \\
\underline{66} \\
40 \\
\underline{33} \\
70 \\
\underline{66} \\
4
\end{array}
\qquad
\begin{array}{r}
2.125 \\
8\,\overline{)17.000} \\
\underline{16}^{\times\times\times} \\
1\ 0 \\
\underline{8} \\
20 \\
\underline{16} \\
40 \\
\underline{40} \\
0
\end{array}
\qquad
\begin{array}{r}
2.142857\ldots \\
7\,\overline{)15.000000} \\
\underline{14}^{\times\times\times\times\times\times} \\
1\ 0 \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
1
\end{array}
\qquad
\begin{array}{r}
.85 \\
20\,\overline{)17.00} \\
\underline{16\ 0}^{\times} \\
1\ 00 \\
\underline{1\ 00} \\
0
\end{array}$$

SAME REMAINDER

Exercise #5, page 45

1. (a) $0.125 = \frac{125}{1000} = \frac{1}{8}$

DIGIT 5 IS IN THE THOUSANDTHS PLACE.

(b) $0.0004 = \frac{4}{10,000} = \frac{1}{2500}$

DIGIT 4 IS IN THE TEN-THOUSANDTHS PLACE.

(c) Let $x = 0.\overline{153} = 0.153153\ldots$

$$\begin{aligned} 1000x &= 153.153\ldots \\ x &= 0.153\ldots \\ \hline 999x &= 153 \quad \text{(sub.)} \\ x &= \frac{153}{999} \\ x &= \frac{17}{111} \end{aligned}$$

(d) Let $x = 0.\overline{7} = 0.777\ldots$

$$\begin{aligned} 10x &= 7.777\ldots \\ x &= 0.777\ldots \\ \hline 9x &= 7 \quad \text{(sub.)} \\ x &= \frac{7}{9} \end{aligned}$$

Exercise #6, page 48

1. (a) $2.\overline{45}$; rational (All repeating decimals are rational.)

(b) 0.2123478...; irrational (All non-terminating, non-repeating decimals are irrational.)

(c) $\frac{5}{9}$; rational (This number is in the rational form $\frac{a}{b}$.)

(d) 0.333...; rational (All repeating decimals are rational.)

(e) 13.675; rational (All terminating decimals are rational.)

(f) 0.325325532555...; irrational (All non-terminating, non-repeating decimals are irrational.)

(g) $\sqrt{3}$; irrational ($\sqrt{3}$ is equivalent to the non-terminating, non-repeating decimal 1.7321... .)

(h) $\sqrt{16}$; rational ($\sqrt{16} = 4$ which is a rational number.)

(i) 0; rational (All integers are rational numbers.)

2. (a) $\sqrt{35} = 5.916$ (b) $\sqrt{84} = 9.165$ (c) $\sqrt{76} = 8.718$ (d) $\sqrt{8} = 2.828$

Exercise #7, page 52

1. (a) true; All natural numbers are integers.
   (b) false; Zero is a whole number but not a natural number.
   (c) true; All integers are rational numbers.
   (d) true; All repeating decimals are rational numbers and hence are real numbers.
   (e) false; π is an irrational number.
   (f) true; $-\sqrt{9} = -3$ which is an integer.
   (g) false; $\frac{4}{5}$ does not belong to the set $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
   (h) false; $\sqrt{10}$ is an irrational number.
   (i) true; 0.2 is equivalent to the rational number $\frac{1}{5}$.
   (j) true; All irrational numbers are real numbers.
   (k) false; $1.\overline{56}$ is a rational number - not an irrational number.
   (l) false; $\sqrt{-2}$ belongs to the set of imaginary numbers.

End of Lesson B

Lesson C

POWERS, ROOTS, VARIABLE EXPRESSIONS

Topic One: Powers

A. Positive Integral Powers

The term POWER refers to a number that can be expressed by using a base and an exponent. The exponent tells us how many times the base is to be used as a factor.

EXAMPLES:

1. 3^4 is a power involving the base 3 and the exponent 4. It tells us that the base 3 is to be used as a factor 4 times.

   i.e. $\underset{\text{BASE}}{3}^{\overset{\text{EXPONENT}}{4}} = \underbrace{3 \times 3 \times 3 \times 3}_{\text{3 IS USED AS A FACTOR 4 TIMES.}} = 81$

   3^4 is read "three to the fourth."

2. $(-7)^3$ is a power involving the base -7 and the exponent 3. It tells us that the base -7 is to be used as a factor 3 times.

   i.e. $(\underset{\text{BASE}}{-7})^{\overset{\text{EXPONENT}}{3}} = \underbrace{-7 \times -7 \times -7}_{\text{-7 IS USED AS A FACTOR 3 TIMES.}} = -343$

   $(-7)^3$ is read "negative seven cubed."

If the base of a power is any real number and its exponent is a positive integer, it is called a POSITIVE INTEGRAL POWER. The following are all examples of positive integral powers.

5^8, $(-2)^4$, $\left(\frac{1}{2}\right)^3$, π^2, $(\sqrt{2})^5$

When writing powers, we always use parentheses around negative bases. This tells us that both the sign and the absolute value of the number are affected by the exponent. For example, the power whose base is -5 and exponent is 4 is written $(-5)^4$. This means that -5 is used as a factor 4 times.

i.e. $(-5)^4 = -5 \times -5 \times -5 \times -5 = 625$

In contrast, -5^4 means that 5 is used as a factor 4 times and a minus sign is placed in front of the result.

i.e. $-5^4 = -(5 \times 5 \times 5 \times 5) = -625$

Similarly, $(-2)^6 =$ ______, while $-2^6 =$ ______.

If the base of a power is a fraction, parentheses must also be used around the base. This tells us that the exponent applies to both the numerator and denominator of the fraction. For example, the power whose base is $\frac{2}{3}$ and exponent is 5 is written $\left(\frac{2}{3}\right)^5$. This means that $\frac{2}{3}$ is used as a factor 5 times.

i.e. $$\left(\frac{2}{3}\right)^5 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{32}{243}$$

In contrast, $\frac{2^5}{3}$ means that 2 used as a factor 5 times and then the result is divided by 3.

i.e. $$\frac{2^5}{3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{3} = \frac{32}{3}$$

Similarly, $\left(\frac{4}{5}\right)^3 =$ ______, while $\frac{4^3}{5} =$ ______.

All positive integral powers whose bases are positive real numbers represent positive numbers. For example, the following powers are all positive.

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$$

$$9^2 = 9 \times 9 = 81$$

$$\left(\frac{1}{4}\right)^4 = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{256}$$

Positive integral powers with negative bases may be either positive or negative. Note that:

1. Any negative real number raised to an odd power represents the product of an odd number of negative factors. Hence, the power represents a negative number.

   EXAMPLES:

   $$(-3)^{\overset{\text{ODD EXPONENT}}{3}} = -3 \times -3 \times -3 = \overset{\text{NEGATIVE NUMBER}}{-27}$$

   $$(-1)^7 = -1 \times -1 \times -1 \times -1 \times -1 \times -1 \times -1 = -1$$

   $$\left(\frac{-1}{2}\right)^5 = \frac{-1}{2} \times \frac{-1}{2} \times \frac{-1}{2} \times \frac{-1}{2} \times \frac{-1}{2} = \frac{-1}{32}$$

   When the base is negative and the exponent is odd, the result is negative.

2. Any negative real number raised to an even power represents the product of an even number of negative factors. Hence, the power represents a positive number.

EXAMPLES:

$$\left.\begin{aligned} &(-8)^{\overset{\text{EVEN EXPONENT}}{2}} = -8 \times -8 = \overset{\text{POSITIVE NUMBER}}{64} \\ &(-1)^6 = -1 \times -1 \times -1 \times -1 \times -1 \times -1 = 1 \\ &\left(\frac{-1}{3}\right)^4 = \frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} = \frac{1}{81} \end{aligned}\right\}$$

When the base is negative and the exponent is even the result is positive.

Self-correcting Exercise #1

Answers may be found on page 53 of this lesson.

1. Fill in the blanks in the following chart.

Power	Base	Exponent	Evaluation
$(-6)^3$	____	____	$-6 \times -6 \times -6 =$ ____
$\left(\frac{5}{8}\right)^2$	____	____	____
$(0.2)^4$	____	____	____
____	-4	4	____
____	____	____	$\frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} =$ ____

2. Express each of the following numbers as a positive integral power with a positive base.

(a) 100,000 = 10^5

(b) 125 = ____

(c) $\frac{4}{9}$ = ____

(d) 121 = ____

(e) $\frac{27}{1000}$ = ____

(f) 128 = ____

3. Express each of the following numbers as a positive integral power with a negative base. (Remember that parentheses must always be used around negative bases.)

(a) $36 = (-6)^2$ (b) $-216 =$ ______ (c) $-32 =$ ______

(d) $100 =$ ______ (e) $\frac{-27}{64} =$ ______ (f) $144 =$ ______

4. State whether the following expressions represent positive or negative numbers.

(a) $(-3)^7$ (b) $(-8)^{16}$ (c) $\left(\frac{-1}{2}\right)^9$

negative ______ ______

(d) -8^4 (e) $(-8)^4$ (f) $-(-5)^{31}$

______ ______ ______

5. Simplify the following numerical expressions. (Evaluate powers before you perform any other arithmetic operations.)

(a) $(-8)^3 + 5^4 = (-8 \times -8 \times -8) + (5 \times 5 \times 5 \times 5) =$ ______ + ______ = ______

(b) $-3(4^3 - 5^2) =$

(c) $\left(\frac{-2}{3}\right)^3 - \frac{5}{9} =$

(d) $\frac{-3 + (-2)^5}{7} =$

(e) $4 - (-3)^2 =$

B. Multiplying and Dividing Positive Integral Powers With Like Bases

The exponent of a positive integral power tells us how many times the base of the power is to be used as a factor. If two powers with like bases are multiplied, the total number of times the base is used as a factor can be found by adding the exponents.

For example,

$$3^3 \times 3^2 = \underbrace{(3 \times 3 \times 3)}_{\text{3 factors}} \times \underbrace{(3 \times 3)}_{\text{2 factors}} = \underbrace{3 \times 3 \times 3 \times 3 \times 3}_{\text{5 factors}} = 3^5 \quad \text{Exponent is 5.}$$

The exponent of the product could have been found by adding the exponents of the two factors.

i.e. $3^3 \times 3^2 = 3^{\overset{\swarrow \text{SUM OF EXPONENTS}}{3+2}} = 3^5$

The PRODUCT OF POWERS PROPERTY tells us that in multiplying powers with the same base, we can keep the common base and add the exponents.

PRODUCT OF POWERS PROPERTY
For any positive integral powers a^m and a^n,
$a^m \times a^n = a^{m+n}$

EXAMPLES:

$$5^8 \times 5^4 = 5^{8+4} = 5^{12}$$

$$(-2)^3 \times \underset{\nwarrow \text{THE EXPONENT OF THIS FACTOR IS 1}}{(-2)} = (-2)^{3+1} = (-2)^4$$

Adding the exponents when multiplying powers is possible only when the bases are the same. For example, to find the product $2^4 \times 3^5$, we cannot add the exponents since the bases are different. This product cannot be written as a single power.

The quotient of two powers with the same base can be expressed as a single power.

For example,

$$\begin{aligned}\frac{10^5}{10^3} &= \frac{\overset{1}{\cancel{10}} \times \overset{1}{\cancel{10}} \times \overset{1}{\cancel{10}} \times 10 \times 10}{\underset{1}{\cancel{10}} \times \underset{1}{\cancel{10}} \times \underset{1}{\cancel{10}}} && \begin{matrix}\leftarrow \text{5 factors} \\ \leftarrow \text{3 factors}\end{matrix} \\ &= \frac{10 \times 10}{1} && \leftarrow \text{2 factors} \\ &= 10^2\end{aligned}$$

The exponent of the quotient could have been found by subtracting the exponent of the denominator from the exponent of the numerator.

$$\text{i.e. } \frac{10^5}{10^3} = 10^{5-3} \xleftarrow{\text{DIFFERENCE OF EXPONENTS}} = 10^2$$

The QUOTIENT OF POWERS PROPERTY tells us that in dividing powers with the same base, we can keep the common base and subtract the exponents.

QUOTIENT OF POWERS PROPERTY

For any positive integral powers a^m and a^n (where $m > n$),

$$\frac{a^m}{a^n} = a^{m-n}$$

EXAMPLES:

$$\frac{3^{12}}{3^{11}} = 3^{12-11} = 3^1 \xleftarrow{\text{EXPONENT 1 CAN BE DROPPED.}} = 3$$

$$\frac{(-8)^{16}}{(-8)^{12}} = (-8)^{16-12} = (-8)^4$$

Again, notice that subtracting the exponents when dividing powers is possible only when the bases are the same. For example, to find the quotient $\frac{10^3}{9^2}$, we cannot subtract the exponents since the bases are different. This quotient cannot be written as a single power.

Also, note that the quotient of powers property applies only when the exponent of the numerator is larger than the exponent of the denominator. This property cannot be extended to cover all cases until we define negative and zero exponents. Therefore, for the time being, a quotient such as $\frac{5^4}{5^{11}}$ will not be considered.

Self-correcting Exercise #2

Answers may be found on page 54 of this lesson.

1. Write each of the following expressions as a single power.

(a) $7^3 \times 7^4 =$

(b) $(-3)\ (-3)\ (-3)^6 =$

(c) $\frac{5^9}{5} =$

(d) $\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 =$

(e) $\frac{(-6)^4\ (-6)^8}{(-6)^7} =$

(f) $\frac{4^{15}}{4^3 \times 4^5} =$

2. Evaluate (Do not leave answers in exponential form.)

(a) $\frac{(-5)^{20}\ (-5)^3}{(-5)^{21}} =$

(b) $\frac{3^{11} \times 3^5 \times 2^{13}}{3^{14} \times 2^9} =$

(c) $\frac{6^8}{6^5} - 6^2 =$

C. Zero Powers and Negative Integral Powers

A ZERO POWER has a base that is a real number and an exponent that is zero. The following are examples of zero powers.

$(-5)^0$, 6^0, $(\sqrt{2})^0$, $\left(\frac{1}{3}\right)^0$, 12^0

A NEGATIVE INTEGRAL POWER has a base that is a real number and an exponent that is a negative integer. The following are examples of negative integral powers.

6^{-3}, $(-7)^{-2}$, $(\sqrt{3})^{-4}$, $\left(\frac{5}{6}\right)^{-1}$, 30^{-5}

You will recall that for any positive integral power, the exponent tells us how many times the base is to be used as a factor. For example, the exponent 3 in the power 2^3 tells us that the base 2 is to be used as a factor 3 times. However, for the zero power 2^0 and the negative integral power 2^{-3}, it is impossible to think of using the base 2 as a factor 0 times or -3 times. Remembering that terms in mathematics mean what we decide they will mean, we define zero powers and negative integral powers in such a way that the new definitions are consistent with previous rules and definitions.

The Product of Powers Property discussed in a previous section was demonstrated only for positive integral powers. We would like to be able to define zero powers and negative integral powers in such a way that we can also apply the Product of Powers Property in finding products like $10^0 \times 10^4$ and $10^{-3} \times 10^5$.

If we could apply the Product of Powers Property when a zero power is involved, we could say that:

$$2^3 \times 2^0 = 2^{3+0} = 2^3$$

Because 1 is the identity element for multiplication, we know already that:

$$2^3 \times 1 = 2^3$$

Thus, if the Product of Powers Property is to hold for zero powers, 2^0 and 1 must be names for the same number.

2^0 MUST EQUAL 1.

$$2^3 \times 2^0 = 2^3$$

$$2^3 \times 1 = 2^3$$

In general, we must define the zero power of any non-zero base as 1. This means that $4^0 = 1$, $(-7)^0 = 1$, $10^0 = 1$, and so on. We restrict this definition to a non-zero base, since 0^0 cannot be defined as 1 without causing ambiguities.

DEFINITION - ZERO EXPONENT

For any real number a, $a \neq 0$,

$$a^0 = 1$$

Thus, $(-14)^0 =$ ____, $2^0 =$ ____, $\left(\frac{-2}{3}\right)^0 =$ ____, and $100^0 =$ ____

If we could apply the Product of Powers Property when a negative integral power is involved, we could say that:

$$2^3 \times 2^{-3} = 2^{3+(-3)} = 2^0 = 1$$

Because 2^3 and $\frac{1}{2^3}$ are multiplicative inverses (or reciprocals) of each other, we know already that:

$$2^3 \times \frac{1}{2^3} = 1 \cdot$$

Thus, if the Product of Powers Property is to hold for negative integral powers, 2^{-3} and $\frac{1}{2^3}$ must be names for the same number.

$\leftarrow 2^{-3}$ MUST EQUAL $\frac{1}{2^3}$.

$$2^3 \times 2^{-3} = 1$$

$$2^3 \times \frac{1}{2^3} = 1$$

In general, we must define the negative integral power of any real number to be the multiplicative inverse (or reciprocal) of the corresponding positive integral power. Thus,

$$5^{-2} = \frac{1}{5^2}$$

$$10^{-4} = \frac{1}{10^4} \qquad \text{and so on.}$$

Negative powers of zero are not defined since this leads to division by zero.

DEFINITION - NEGATIVE EXPONENTS

For any negative integral power a^{-m}, $a \neq 0$,

$$a^{-m} = \frac{1}{a^{m}}$$

Use this definition to fill in the blanks below:

1. $5^{-3} = \dfrac{1}{5^{\underline{\quad}}}$

2. $(-2)^{-7} = \dfrac{1}{(-2)^{\underline{\quad}}}$

3. $\dfrac{1}{8^{-6}} = \dfrac{1}{\dfrac{1}{8^{\underline{\quad}}}} = 1 \div \dfrac{1}{8^{6}} = 1 \times \dfrac{8^{6}}{1} = 8^{\underline{\quad}}$

4. $\dfrac{1}{4 \times 3^{-2}} = \dfrac{1}{4 \times \dfrac{1}{3^{2}}} = 1 \div \dfrac{4}{3^{2}} = 1 \times \dfrac{3^{2}}{4} = \dfrac{3^{\underline{\quad}}}{4}$

Now it is easy to interpret the quotient of two powers with like bases when the exponent in the numerator is smaller than the exponent in the denominator. For example,

$$\frac{10^{3}}{10^{5}} = 10^{3-5} = 10^{-2}$$

Since $10^{-2} = \frac{1}{10^{2}}$, the statement is consistent with the following procedure.

$$\frac{10^{3}}{10^{5}} = \frac{\cancel{10} \times \cancel{10} \times \cancel{10}}{\cancel{10} \times \cancel{10} \times \cancel{10} \times 10 \times 10} = \frac{1}{10^{2}} \text{ (or) } 10^{-2}$$

Thus, when dividing powers with like bases, we may subtract the exponents in either way.

EXAMPLES:

$$\frac{5^{8}}{5^{12}} = 5^{8-12} = 5^{-4}$$

(or)

$$\frac{5^{8}}{5^{12}} = \frac{1}{5^{12-8}} = \frac{1}{5^{4}}$$

RESULTS ARE EQUAL.

$$\frac{3^{-5}}{3^{2}} = 3^{-5-2} = 3^{-7}$$

(or)

$$\frac{3^{-5}}{3^{2}} = \frac{1}{3^{2-(-5)}} = \frac{1}{3^{7}}$$

RESULTS ARE EQUAL.

Zero powers and negative integral powers have been defined in such a way that the Product of Powers Property and Quotient of Powers Property are satisfied for all powers with integral exponents. Thus,

$$3^8 \times 3^{-5} = 3^{8+(-5)} = 3^3$$

$$6^{-7} \times 6^{-1} = 6^{(-7)+(-1)} = 6^{-8} \qquad \left(\text{or } \frac{1}{6^8}\right)$$

$$\frac{4^{-5}}{4^{-4}} = 4^{-5-(-4)} = 4^{-1} \qquad \left(\text{or } \frac{1}{4}\right)$$

$$\frac{5^{-2}}{5^{-6}} = 5^{-2-(-6)} = 5^4$$

Self-correcting Exercise #3

Answers may be found on page 54 of this lesson.

1. Express each of the following numbers as a power of 3.

   (a) $243 = 3^{\underline{\quad}}$ (b) $1 = 3^{\underline{\quad}}$

   (c) $\frac{1}{27} = 3^{\underline{\quad}}$ (d) $\frac{1}{9} = 3^{\underline{\quad}}$

   (e) $9 = 3^{\underline{\quad}}$ (f) $\frac{1}{81} = 3^{\underline{\quad}}$

2. Evaluate each of the following powers.

   (a) $4^0 =$ (b) $(-1)^3 =$

   (c) $4^{-3} =$ (d) $(-5)^{-2} =$

   (e) $(0.7)^2 =$ (f) $\left(\frac{1}{2}\right)^{-4} =$

   (g) $4 \times 3^{-1} =$

   (h) $2^{-3} \times 4^2 =$

   (i) $\left(\frac{3}{4}\right)^{-3} \times 9^2 =$

   (j) $\frac{6}{8^{-2}} =$

3. Write these expressions with positive exponents.

(a) $6^{-8} =$

(b) $\frac{1}{4^{-6}} =$

(c) $3 \times 5^{-7} =$

(d) $\frac{1}{2^{-9}} + 3^{-5}$

(e) $\frac{7^{-5}}{4^{-8}} =$

4. Use the Power Properties to write each expression as a single power.

(a) $6^{18} \times 6^{-6} =$

(b) $\frac{4^{8}}{4^{-2}} =$

(c) $\frac{3^{-9}}{3^{-3}} =$

(d) $4^{-5} \times 4^{-7} \times 4^{3} =$

(e) $\frac{2^{-4} \times 2^{-3}}{2^{5}} =$

EXERCISE - Powers

1. Fill in the blanks.

(a) The power $(-7)^5$ means that the ____________, which is -7, must be used 5 times as a ____________.

(b) $\left(\frac{-2}{3}\right)^6$ is a ____________ ____________ power since its ____________ is a real number and its exponent is a ____________ integer.

(c) When writing powers, parentheses must always be used around ____________ or ____________ bases.

(d) A power represents a negative number if its base is ____________ and its exponent is an ____________ number.

(e) The Product of Powers Property tells us that in multiplying powers with ____________ bases, we keep the common ____________ and add the ____________.

(f) Any non-zero real number whose exponent is ____________ is equal to one.

(g) 2^{-13} is defined to be the ____________ of 2^{13}.

(h) When dividing powers with like bases we ____________ the exponents.

2. Express each of the following numbers as a power of 2.

(a) $\frac{1}{64}$ = 2^{-6}

(b) 4 = ______

(c) 32 = ______

(d) $\frac{1}{2}$ = ______

(e) 1 = ______

(f) $\frac{1}{8}$ = ______

3. Express each of the following numbers as a power of 5.

(a) $25 = \underline{5^2}$ (b) $\frac{1}{5} =$ ______ (c) $\frac{1}{125} =$ ______

(d) $625 =$ ______ (e) $\frac{1}{25} =$ ______ (f) $1 =$ ______

4. Write each expression with positive exponents only. (Do not evaluate.)

(a) $5^{-8} =$

(b) $4^{-6} \times 6^7 =$

(c) $\frac{1}{3^{-9}} = \frac{1}{\frac{1}{3^9}} =$

(d) $\frac{7}{4^{-3}} =$

(e) $\frac{10^{-7}}{12^{-8}} =$

(f) $6^{-8} + 4^{-2} =$

5. Evaluate the following expressions. (Do not leave in exponential form.)

(a) $(-4)^3 =$ (b) $\left(\frac{-1}{2}\right)^5 =$

(c) $(0.02)^2 =$ (d) $(1.7)^0 =$

(e) $8^{-2} =$ (f) $(-3)^{-4} =$

(g) $\frac{1}{3^{-3}} =$ (h) $\left(\frac{2}{3}\right)^{-1} =$

(i) $\frac{6^{18}}{6^{15}} =$

5. (continued) Evaluate. (Do not leave in exponential form.)

(j) $(-8)^0 \ (-8)^2 =$

(k) $\frac{5^{10}}{5^9 \times 5} =$

(l) $(-1)^{35} \times \frac{1}{2^{-2}} =$

(m) $\frac{4^0}{4^{-2}} =$

(n) $\frac{2^2}{3} - \left(\frac{-2}{3}\right)^2 =$

(o) $(-3)^3 - 2^2 =$

(p) $-2(3^2 - 5^2) =$

6. Write each of the following expressions as a single power.

(a) $\frac{4^6}{4^{15}} = \frac{1}{4^9}$

(b) $6^{-5} \times 6^{-11} =$

(c) $\frac{(\sqrt{5})^7}{(\sqrt{5})^6} =$

(d) $\frac{(-8)^{14}}{(-8)^4} =$

(e) $4^5 \times 4^{-9} =$

(f) $\frac{3^3}{3^{-7}} =$

(g) $\frac{5^3 \times 5^5}{5^2} =$

(h) $2^0 \times 2^5 \times 2^{-12} =$

(i) $\frac{(-3)^6 \ (-3)^8}{(-3)^{-4} \ (-3)^7} =$

(j) $\frac{3 \times 3^{-4}}{3^6} =$

Topic Two: Roots of Real Numbers

A. Meaning of Square Root

The important facts about square roots of real numbers are summarized below.

1. The operation of finding a square root involves finding a number which when multiplied by itself equals a given number. For example, finding the square root of 9 involves finding a number which when multiplied by itself equals 9.

2. When a number is expressed as the product of two factors that are equal, each factor is a square root of the given number. For example, $\frac{4}{5}$ must be a square root of $\frac{16}{25}$ since

$$\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}.$$

3. Every positive real number has two square roots that are equal in absolute value but opposite in sign. The positive square root of a number is called its PRINCIPAL SQUARE ROOT and is represented by the symbol $\sqrt{}$. The symbol $-\sqrt{}$ is used to represent the negative square root of a number. For example, the positive real number 5 has two square roots. $\sqrt{5}$ represents the principal square root of 5 and $-\sqrt{5}$ represents the negative square root of 5.

4. A negative real number has no square root in the system of real numbers since we cannot find two equal numbers that will give a negative product. For example, $\sqrt{-3}$ is not a real number. Numbers of this type are called IMAGINARY NUMBERS. (They will be covered later in this course.)

5. The square root of a positive number may be either rational or irrational. Any number that has a rational square root is a PERFECT SQUARE. For example, 0.36 is a perfect square since $\sqrt{0.36} = 0.6$ and $-\sqrt{0.36} = -0.6$ which are rational numbers.

6. Some square roots represent irrational numbers. For example $\sqrt{2}$ represents the same irrational number as the non-terminating, non-repeating decimal 1.4142136... .

7. Zero has only one square root. The square root of 0 is 0.

8. $\sqrt{x}$ is called a SECOND-ORDER RADICAL. x is called the RADICAND. For example, $\sqrt{13}$ is a second-order radical with a radicand of 13.

Self-correcting Exercise #4

Answers may be found on page 56 of this lesson.

1. Fill in the blanks.

(a) $\sqrt{144}$ = ________ (b) $-\sqrt{144}$ = ________

(c) $(\sqrt{0.9})^2$ ________ (d) $\sqrt{\frac{4}{25}}$ = ________

(e) $\sqrt{6} \times \sqrt{6}$ = ________ (f) $-\sqrt{3} \times -\sqrt{3}$ = ________

(g) $-\sqrt{81}$ = ________ (h) $\sqrt{3600}$ = ________

2. Decide whether each number is rational, irrational, or imaginary.

(a) $\sqrt{17}$ (b) $-\sqrt{36}$ (c) $\sqrt{-36}$

irrational ________ ________

(d) $\sqrt{\frac{81}{100}}$ (e) $\sqrt{\frac{2}{5}}$ (f) $-\sqrt{-2}$

________ ________ ________

B. Multiplying Second-Order Radicals

Can we find a rule for writing the product of two second-order radicals as a single radical?

Consider the product $\sqrt{16} \times \sqrt{25}$. We know that $\sqrt{16} = 4$ and $\sqrt{25} = 5$. Thus,

$$\sqrt{16} \times \sqrt{25} = 4 \times 5 = 20 .$$

Since 20 is the principal square root of 400, we could write the answer in the form $\sqrt{400}$. This same result could be obtained by multiplying the radicands and taking the square root of the result.

$$\text{i.e. } \sqrt{16} \times \sqrt{25} = \underbrace{\sqrt{16 \times 25}}_{\uparrow\ \text{PRODUCT OF RADICANDS}} = \sqrt{400} = 20$$

In general, the product of any second order radicals can be found by multiplying the radicands and taking the square root of the result.

PRODUCT OF SECOND-ORDER RADICALS

For any positive real numbers a and b,

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

PRODUCT OF RADICANDS

EXAMPLES: $\sqrt{3} \times \sqrt{5} = \sqrt{3 \times 5} = \sqrt{15}$

$\sqrt{7} \times \sqrt{2} = \sqrt{7 \times 2} = \sqrt{14}$

$\sqrt{6} \times \sqrt{13} = \sqrt{6 \times 13} = \sqrt{78}$

An interesting result occurs when you take the product of two identical radicals. The result is always a rational number that is the same as the radicand of each radical.

EXAMPLES: $\sqrt{2} \times \sqrt{2} = \sqrt{2 \times 2} = \sqrt{4} = 2$

$\sqrt{7} \times \sqrt{7} = \sqrt{7 \times 7} = \sqrt{49} = 7$

$\sqrt{5} \times \sqrt{5} = \sqrt{5 \times 5} = \sqrt{25} = 5$

*

PRODUCT OF IDENTICAL SECOND-ORDER RADICALS

For any positive real number a,

$$\sqrt{a} \times \sqrt{a} = a$$

Thus, when finding the product of identical radicals, you can write down the answer immediately. That is,

$$\left.\begin{array}{l} \sqrt{10} \times \sqrt{10} = 10 \\ \sqrt{3} \times \sqrt{3} = 3 \end{array}\right\} \text{Product corresponds to the radicand.}$$

A MIXED RADICAL is the product of a rational number and a radical. For example, $-3\sqrt{2}$ is a mixed radical. It represents the product of the rational number -3 and the radical $\sqrt{2}$.

The product of mixed radicals can be determined by finding the product of the rational parts and multiplying this result by the product of the radicals. The commutative and associative properties of multiplication allow you to rearrange and group factors in this manner.

PRODUCT OF SECOND-ORDER MIXED RADICALS

For any positive real numbers a and b and rational numbers c and d,

$$c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$$

EXAMPLE: $-2\sqrt{2} \times 3\sqrt{5} =$ ▨

Solution

Rearrange the factors so that the rational parts are grouped together and the radicals are grouped together.

$$-2\sqrt{2} \times 3\sqrt{5} = (-2 \times 3)(\sqrt{2} \times \sqrt{5})$$

Then, apply the rule for multiplying radicals.

$$= \underline{\underline{-6\sqrt{10}}}$$

Self-correcting Exercise #5

Answers may be found on page 56 of this lesson.

1. Write the following expressions as single radicals or mixed radicals.

(a) $\sqrt{5} \times \sqrt{7} =$ ________

(b) $-\sqrt{3} \times \sqrt{7} =$ ________

(c) $\sqrt{13} \times \sqrt{13} =$ ________

(d) $-2\sqrt{2} \times -\sqrt{3} =$ ________

(e) $(\sqrt{5})^2 =$ ________

(f) $-2\sqrt{3} \times 3\sqrt{5} =$ ________

(g) $\sqrt{11} \times \sqrt{5} \times \sqrt{2} =$ ________

(h) $3\sqrt{2} \times -2\sqrt{5} \times 5\sqrt{3} =$ ________

(i) $(-\sqrt{10})^2 =$ ________

(i) $5\sqrt{\frac{2}{3}} \times 3\sqrt{\frac{3}{5}} =$ ________

C. Dividing Second-Order Radicals

Can we find a rule for writing the quotient of two radicals as a single radical?

Consider the quotient $\frac{\sqrt{36}}{\sqrt{9}}$. We know that $\sqrt{36} = 6$ and $\sqrt{9} = 3$.

Thus,

$$\frac{\sqrt{36}}{\sqrt{9}} = \frac{6}{3} = 2$$

Since 2 is the principal square root of 4, we could write the answer in the form $\sqrt{4}$. The same result could be obtained by dividing the radicands and taking the square root of the result.

In general, the quotient of two second-order radicals can be written as a single radical by dividing the radicands and taking the square root of the result.

i.e. $$\frac{\sqrt{36}}{\sqrt{9}} = \overset{\text{QUOTIENT OF RADICANDS}}{\sqrt{\frac{36}{9}}} = \sqrt{4}$$

QUOTIENT OF SECOND-ORDER RADICALS

For any positive real numbers a and b,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}\ .$$

EXAMPLES:

$$\frac{\sqrt{30}}{\sqrt{6}} = \sqrt{\frac{30}{6}} = \sqrt{5}$$

$$\frac{\sqrt{35}}{\sqrt{5}} = \sqrt{\frac{35}{5}} = \sqrt{7}$$

$$\frac{\sqrt{51}}{\sqrt{17}} = \sqrt{\frac{51}{17}} = \sqrt{3}$$

The quotient of mixed radicals can be determined by finding the quotient of the rational parts and multiplying this result by the quotient of the radicals.

QUOTIENT OF SECOND-ORDER MIXED RADICALS

For any positive real numbers a and b and rational numbers c and d,

$$\frac{c\sqrt{a}}{d\sqrt{b}} = \frac{c}{d}\sqrt{\frac{a}{b}}\ .$$

EXAMPLE: $\frac{6\sqrt{33}}{-2\sqrt{11}} =$ ▨

Solution

$$\frac{6\sqrt{33}}{-2\sqrt{11}} = \underset{\text{QUOTIENT OF RATIONAL PARTS}}{\frac{6}{-2}} \times \underset{\text{QUOTIENT OF RADICALS}}{\frac{\sqrt{33}}{\sqrt{11}}} = -3\sqrt{\frac{33}{11}} = \underline{\underline{-3\sqrt{3}}}$$

When two radicals are expressed as a quotient, you will sometimes find that the radicand in the denominator is not a factor of the radicand in the numerator. In such cases, there is little advantage to applying the rule for dividing radicals. Instead, we use a process called RATIONALIZING THE DENOMINATOR. Using this process, we multiply both radicals in the quotient by a radical that is identical to the divisor. (This will not alter the quotient since we have multiplied both terms of the fraction by the same number.) Since the denominator is now expressed as the product of two identical radicals, it is now a rational number.

For example, consider the quotient $\frac{2\sqrt{3}}{\sqrt{5}}$. We could apply the rule for dividing radicals and express this quotient in the form $2\sqrt{\frac{3}{5}}$. But, this form is really no simpler than the original. Instead, we can make the denominator of the fraction $\frac{2\sqrt{3}}{\sqrt{5}}$ a rational number by multiplying both terms of the fraction by $\sqrt{5}$.

i.e. $$\frac{2\sqrt{3}}{\sqrt{5}} = \frac{2\sqrt{3}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \leftarrow \text{THIS IS THE SAME AS MULTIPLYING BY 1.}$$

$$= \frac{2\sqrt{15}}{5} \leftarrow \text{DENOMINATOR IS NOW A RATIONAL NUMBER.}$$

Self-correcting Exercise #6

Answers may be found on page 56 of this lesson.

1. Simplify the following quotients. If the radicand in the denominator is a factor of the radicand in the numerator, apply the rule for dividing radicals. Otherwise, simplify the quotient by rationalizing the denominator.

   (a) $\frac{2\sqrt{7}}{\sqrt{3}} =$

   (b) $\frac{-27\sqrt{6}}{9\sqrt{3}} =$

   (c) $\frac{-8\sqrt{42}}{-2\sqrt{7}} =$

   (d) $\frac{-2}{\sqrt{15}} =$

D. Adding and Subtracting Like Mixed Radicals of the Second Order

Mixed radicals of the second order are said to be LIKE TERMS if they have the same radical. For example, $3\sqrt{2}$ and $-5\sqrt{2}$ are like terms since the radical $\sqrt{2}$ is involved in both terms.

You will recall that multiplication distributes over both addition and subtraction in set R. If we apply the distributive property in reverse, we can use it to add or subtract like mixed radicals.

EXAMPLE:

$$3\sqrt{2} - 5\sqrt{2} + 7\sqrt{2}$$

$$= (3 - 5 + 7)\sqrt{2}$$

$$= \underline{\underline{5\sqrt{2}}}$$

In effect, we have added and subtracted the coefficients of the mixed radicals and multiplied the result by the common radical.

SUM OR DIFFERENCE OF SECOND-ORDER MIXED RADICALS

For any positive real numbers a and b and rational numbers c and d,

$$c\sqrt{a} + d\sqrt{a} = (c + d)\sqrt{a}$$

$$c\sqrt{a} - d\sqrt{a} = (c - d)\sqrt{a}$$

Self-correcting Exercise #7

Answers may be found on page 57 of this lesson.

1. In each group of mixed radicals, circle the like terms.

(a) $\sqrt{2}$, $3\sqrt{2}$, $3\sqrt{5}$ $2\sqrt{3}$ (b) $\sqrt{11}$, $11\sqrt{3}$, $2\sqrt{11}$, $-\sqrt{11}$

(c) $6\sqrt{5}$, $5\sqrt{7}$, $3\sqrt{7}$, $5\sqrt{6}$ (d) $2\sqrt{10}$, $7\sqrt{6}$, $10\sqrt{2}$, $-\sqrt{6}$, $\sqrt{-6}$

2. Simplify the following sums and differences.

(a) $5\sqrt{6} + 9\sqrt{6} =$ (b) $-2\sqrt{3} + 7\sqrt{3} =$

(c) $-6\sqrt{5} - 8\sqrt{5} =$ (d) $4\sqrt{7} - \sqrt{7} =$

(e) $3\sqrt{2} + 7\sqrt{2} - 12\sqrt{2}$ (f) $2\sqrt{5} + 3\sqrt{2} - \sqrt{5} - 6\sqrt{2}$

$=$ $=$

E. Simplifying Radicals

A second-order radical is in simplest form when:

(1) The radicand contains no factor that is a perfect square.
(2) The radicand is not a fraction.

The integers 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, ... are perfect squares. You must try dividing a given radicand by these perfect square factors in order to see if it is in simplest form. If a radicand contains a perfect square factor, this factor can be removed from under the root sign by applying the rule, in reverse, for multiplying radicals.

EXAMPLE: Write $\sqrt{125}$ in simplest form.

Solution

The perfect square factor 25 divides into the radicand 125. We must express 125 as the product of this perfect square factor, 25, and the remaining factor, 5.

$$\sqrt{125} = \sqrt{25 \times 5}$$

The rule for multiplying radicals, written in reverse, tells us that $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

Thus,

$$\begin{aligned}\sqrt{125} &= \sqrt{25} \times \sqrt{5} \\ &= 5\sqrt{5}\end{aligned}$$

Thus, the radical $\sqrt{125}$ can be written in simplest form as the mixed radical $5\sqrt{5}$.

By removing perfect square factors from the radicals involved in a sum or difference, you can sometimes uncover like terms. For example in the sum $(-5\sqrt{48} + 8\sqrt{3} + 2\sqrt{147})$, it does not appear that there are any like terms. But, by removing perfect square factors from the radicals $\sqrt{48}$ and $\sqrt{147}$, you will find that all three terms are alike.

i.e.
$$\begin{aligned}-5\sqrt{48} + 8\sqrt{3} + 2\sqrt{147} &= -5\sqrt{\overset{\swarrow\text{PERFECT SQUARE}}{\textcircled{16}} \times 3} + 8\sqrt{3} + 2\sqrt{\overset{\swarrow\text{PERFECT SQUARE}}{\textcircled{49}} \times 3} \\ &= (-5\sqrt{16}\ \sqrt{3}) + 8\sqrt{3} + (2\sqrt{49}\ \sqrt{3}) \\ &= (-5 \times 4\sqrt{3}) + 8\sqrt{3} + (2 \times 7\sqrt{3}) \\ &= -20\sqrt{3} + 8\sqrt{3} + 14\sqrt{3} \\ &= (-20 + 8 + 14)\sqrt{3} \\ &= \underline{\underline{2\sqrt{3}}}\end{aligned}$$

If a radicand is a fraction, the radical can be written in simplest form by first expressing it as a quotient of radicals and then rationalizing the denominator.

EXAMPLE: Write $\sqrt{\frac{2}{5}}$ in simplest form.

Solution

The rule for dividing radicals, written in reverse, tells us that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

Thus,

$$\sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}}$$

Next, we can rationalize the denominator by multiplying both terms of the fraction by $\sqrt{5}$.

$$= \frac{\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{\sqrt{10}}{5}$$

Thus, the radical $\sqrt{\frac{2}{5}}$ can be written in simplest form as $\frac{\sqrt{10}}{5}$.

Self-correcting Exercise #8

Answers may be found on page 57 of this lesson.

1. Write in simplest form by removing perfect square factors.

   (a) $\sqrt{32} =$

   (b) $\sqrt{27} =$

   (c) $\sqrt{700} =$

   (d) $\sqrt{243} =$

2. Write in simplest form by rationalizing the denominators.

   (a) $\sqrt{\frac{1}{2}} =$

   (b) $\sqrt{\frac{2}{3}} =$

   (c) $-3\sqrt{\frac{2}{5}} =$

3. Perform the following operations. All answers should be expressed in lowest terms.

(a) $3\sqrt{2} \times 5\sqrt{10} =$

(b) $3\sqrt{28} - 5\sqrt{63} =$

(c) $\frac{5\sqrt{6}}{3\sqrt{2}} =$

(d) $2\sqrt{75} - 3\sqrt{108} + \sqrt{147} =$

EXERCISE - Roots of Real Numbers

1. Fill in the blanks.

(a) Every ______________ real number has two square roots that are equal in ______________ value but opposite in _____________.

(b) $\sqrt{3}$ represents the _______________________ square root of 3 while $-\sqrt{3}$ represents the ______________________ square root of 3.

(c) Since $\sqrt{121}$ is a rational number, 121 is called a ________________ _____________.

(d) The product of two second-order radicals can be found by multiplying the ______________ and taking the ___________________ _____________ of the result.

(e) The product of ______________ second-order radicals is always a rational number.

(f) The square root of any _________________ real number does not belong to set R.

(g) $2\sqrt{3}$ is called a _______________ _________________ because it is the product of a rational number and a radical.

(h) In order to write the quotient $\frac{3}{\sqrt{5}}$ in simplest form, we must

__________ the denominator. This can be done by multiplying

both terms by ______.

2. Write each number as the product of two factors, one of which is the largest possible perfect square factor.

PERFECT SQUARE

(a) $300 = \underline{100} \times \underline{3}$ (b) $160 = \_\_\_\_ \times \_\_\_\_$

(c) $44 = \_\_\_\_ \times \_\_\_\_$ (d) $128 = \_\_\_\_ \times \_\_\_\_$

(e) $245 = \_\_\_\_ \times \_\_\_\_$ (f) $363 = \_\_\_\_ \times \_\_\_\_$

3. Express the following radicals in simplest form. (Make sure that you remove all perfect square factors from under the root signs.)

(a) $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$

(b) $\sqrt{162} =$

(c) $\sqrt{1000} =$

(d) $\sqrt{\frac{1}{8}} =$

(e) $\sqrt{\frac{3}{2}} =$

(f) $\frac{6}{5\sqrt{12}} =$

(g) $-2\sqrt{\frac{4}{5}} =$

(h) $-3\sqrt{80} =$

4. Perform the following operations. Answers should be expressed in simplest form. (Review discussion of "simplest form" at the top of p. 23.)

(a) $\sqrt{5} \times -\sqrt{5} =$

(b) $\sqrt{2} \times \sqrt{10} =$

(c) $(-\sqrt{6})^2 =$

4. (continued) Write all answers in simplest form.

(d) $(2\sqrt{3})^2 =$

(e) $4\sqrt{3} \times -3\sqrt{15} =$

(f) $2\sqrt{\frac{3}{5}} \times 5\sqrt{\frac{5}{3}} =$

(g) $-\sqrt{3} \times -\sqrt{2} \times -\sqrt{6} =$

(h) $-3\sqrt{2} \times 4\sqrt{2} \times \sqrt{7} =$

(i) $\frac{-1}{3\sqrt{5}} =$

(j) $\sqrt{\frac{3}{4}} =$

(Hint: $\sqrt{4} = 2$)

(k) $\frac{15\sqrt{10}}{5\sqrt{2}} =$

(Hint: Review example at the bottom of p.20.)

(l) $\frac{2\sqrt{6}}{\sqrt{10}} =$

(m) $\sqrt{\frac{2}{3}} + 5\sqrt{\frac{2}{3}} =$

(Hint: Use distributive property first. Then write $\sqrt{\frac{2}{3}}$ in simplest form.)

(n) $3\sqrt{13} + 4\sqrt{13} - 9\sqrt{13}$

$=$

(o) $\sqrt{3} - 4\sqrt{48} + 3\sqrt{75}$

$= \sqrt{3} - 4\sqrt{16 \times 3} + 3\sqrt{25 \times 3}$

$= \sqrt{3} - 4(4\sqrt{3}) + 3(5\sqrt{3})$

$=$

(p) $-5\sqrt{44} - \sqrt{99} + 6\sqrt{11}$

$=$

(q) $10\sqrt{28} + 4\sqrt{63} - 5\sqrt{112}$

$=$

Topic Three: Operating With Variable Terms

A. Variable Expressions

A VARIABLE is a letter symbol that may represent any of the elements of a specified set. This set is called the DOMAIN of the variable. The individual members of the domain are called the VALUES OF THE VARIABLE.

A NUMERICAL EXPRESSION is a statement involving numbers and operation symbols. For example, $5(-6 + 8)$ is a numerical expression involving the numbers 5, -6, and 8, and the operations of multiplication and addition. If these operations are performed on the numbers as indicated, the expression can be evaluated.

i.e. $5(-6 + 8) = 5(2) = \underline{10}$

This numerical expression represents the number 10.

A VARIABLE EXPRESSION (or ALGEBRAIC EXPRESSION) is a statement involving numbers, variables, and operation symbols. A variable expression tells us what operations are to be performed on a given number or numbers, but it does not take on a numerical value until each variable is replaced by a specific value from its domain. Once each variable is assigned a specific value, all indicated operations can be performed, and a numerical value for the expression can be determined. For example, $3(x - 8)$ is a variable expression involving the numbers 3 and 8, the variable x, and the operations of multiplication and subtraction. It instructs us to decrease a number by 8 and multiply the result by 3. Suppose that the number -2 belongs to the domain of x. If we substitute -2 for x, we can evaluate the expression for this particular value from the domain.

i.e. when $x = -2$, $3(x - 8) = 3(-2 - 8) = 3(-10) = \underline{-30}$

Some algebraic expressions involve more than one variable. Each variable involved must be replaced by a specific value from its domain before the expression can be evaluated.

EXAMPLE: Evaluate $5a^2 - bc + 2c^2$ when $a = 2$, $b = -3$, and $c = -4$.

Solution

Substitute 2 for a, -3 for b, and -4 for c.
(Remember the rules for adding, subtracting, and multiplying integers.)

$$\begin{aligned} 5a^2 - bc + 2c^2 &= 5(2)^2 - (-3)(-4) + 2(-4)^2 \\ &= 5(4) - (12) + 2(16) \\ &= 20 - 12 + 32 \\ &= 52 - 12 \\ &= \underline{40} \end{aligned}$$

If the domain of a variable is not specified, it is assumed to be R, the set of real numbers. Since the domain is R or a subset of R, variable expressions represent real numbers. Consequently, all the properties of the real number system as outlined on page 49, lesson B, can be applied to operations with variable expressions.

Self-correcting Exercise #9

Answers may be found on page 58 of this lesson.

1. Fill in the blanks in the following statements.

(a) $-5x + 7$ tells us to ________________ a number by -5 and ___________ 7 to the result.

(b) $\frac{y^2}{2}$ tells us to ________________ a number and ________________ the result by _______.

(c) $\frac{2}{3}(m - 5)$ tells us to ________________ 5 from a number and ___________ the result by _______.

(d) $\sqrt{3x}$ tells us to ________________ a number by ______ and take the _______________ square root of the result.

(e) $(2y - 1)^2$ tells us to ________________ a number by ______, subtract 1, and then ________________ the result.

2. Evaluate the following expressions for the values of the variables that are specified.

(a) $-5y^3$ when $y = -2$
$-5y^3 = -5(-2)^3 =$

(b) $3(x + 1)^2$ when $x = -5$
$3(x + 1)^2 =$

(c) $y^2 - 5x$ when $x = 2$ and $y = 3$
$y^2 - 5x =$

(d) $3a^2 - 2ab + b^2$ when $a = 2$ and $b = -1$
$3a^2 - 2ab + b^2 =$

(e) $3m^2n - 5n^2$ when $m = 2$ and $n = -3$
$3m^2n - 5n^2 =$

B. Adding and Subtracting Variable Terms

An ALGEBRAIC TERM is always written as the product of a real number and one or more variables. The number factor is called the NUMERICAL COEFFICIENT of the algebraic term while the variable factor is called the LITERAL COEFFICIENT.

EXAMPLES:

1. $\frac{-2}{3}x^2y$ is an algebraic term that is the product of the real number $\frac{-2}{3}$ and the variable factors x, x, and y. The numerical coefficient of this term is $\frac{-2}{3}$ and the literal coefficient is x^2y.

2. $\frac{5ab}{c}$ is an algebraic term that is the product of the real number 5 and the variable factors a, b, and $\frac{1}{c}$. The numerical coefficient of this term is 5 and the literal coefficient is $\frac{ab}{c}$.

Some algebraic terms have no visible numerical factor. Such terms have numerical coefficients of 1 or -1.

EXAMPLES:

1. ab^2c is an algebraic term with a numerical coefficient of 1 and a literal coefficient of ab^2c.

2. $-y$ is an algebraic term with a numerical coefficient of -1 and a literal coefficient of y.

Algebraic terms with the same literal coefficient are called LIKE TERMS. For example, 3mn and $\frac{-mn}{2}$ are like terms since they both have a literal coefficient of mn.

Algebraic terms can be added or subtracted if they are like terms. The distributive property applied in reverse allows us to add or subtract like terms.

EXAMPLE:

$$\begin{aligned} &3a^2b - 15a^2b + 9a^2b \\ =\ &(3 - 15 + 9)\ a^2b \\ =\ &\underline{-3a^2b} \end{aligned}$$

In effect, we have added and subtracted the numerical coefficients and multiplied the result by the common literal coefficient.

SUM OR DIFFERENCE OF LIKE ALGEBRAIC TERMS

For any real numbers a and b and variable x,

$$ax + bx = (a + b)x$$
$$ax - bx = (a - b)x$$

Some algebraic expressions can be simplified by combining like terms. If any differences of terms are thought of as sums, the commutative and associative properties of addition can be used to change the order and manner of grouping terms so that like terms can be grouped together. (Remember that addition is commutative and associative while subtraction is not.) Then, like terms can be combined.

EXAMPLE: $4x^2 + 9x - 3y - 12x + x^2 - 5y$

Think of each difference as being a sum.

$= 4x^2 + 9x + (-3y) + (-12x) + x^2 + (-5y) \leftarrow$ (THINK THIS STEP. DON'T WRITE IT.)

The commutative and associative properties of addition can now be used to group terms with the same literal coefficient.

$= (4x^2 + x^2) + (9x - 12x) + (-3y - 5y)$

Combine like terms.

$= \underline{\underline{5x^2 - 3x - 8y}}$

Self-correcting Exercise #10

Answers may be found on page 58 of this lesson.

1. Give the numerical coefficient and literal coefficient of each term.

	Numerical Coefficient	Literal Coefficient
(a) $3r^2s$	________	________
(b) $-b^2$	________	________
(c) $\frac{mn}{3s}$	________	________
(d) $\frac{-6}{y}$	________	________
(e) $\frac{ab}{c}$	________	________
(f) $\frac{2y}{3x}$	________	________

2. Write each algebraic term as the product of a numerical factor and literal factors with exponents of 1.

(a) $\frac{ab^2}{3c^2} = \frac{1}{3} \times a \times b \times b \times \frac{1}{c} \times \frac{1}{c}$

(b) $-6x^2y^2 =$

(c) $\frac{mn^3}{5} =$

(d) $\frac{3a^2bc}{7} =$

(e) $-y^4z =$

3. Simplify the following expressions by combining like terms.

(a) $3mn + 2mn =$ ________

(b) $6a^2 - 5a^2 =$ ________

(c) $3b - 7b =$ ________

(d) $-2xy^2 - 4xy^2$ ________

(e) $4abc - 4abc =$ ________

(f) $-6m + 3m =$ ________

(g) $\sqrt{2}y + 5\sqrt{2}y =$ ________

(h) $\sqrt{3}y - 7\sqrt{3}y =$ ________

(i) $3a + 2b - b + 4a$

=

(j) $x^2 - xy + y^2 - 3y^2 - xy + 4x^2$

=

(k) $3m + 4n - 5m - n + 2n^2$

=

(l) $xy - y^2 + y - 3xy + 3y^2 - 4y$

=

(m) $5x^2 - 6x + 8 - 8x^2 + 7x + 11 + 2x^2 - 3x - 4$

=

C. Multiplying Variable Terms

The Product of Powers Property tells us that in order to multiply two powers with like bases, we keep the common base and add the exponents. That is, for any integral powers a^m and a^n ,

$$a^m \times a^n = a^{m+n}$$

Since all variables represent real numbers, this property can also be applied when the base of a power is a variable rather than a real number.

EXAMPLES:

$$\left.\begin{aligned} x^3 \times x^4 &= x^{3+4} = x^7 \\ y^2 \times y &= y^{2+1} = y^3 \\ m^{-2} \times m^5 &= m^{-2+5} = m^3 \end{aligned}\right\} \quad \text{In each case, we have kept the common base and added the exponents.}$$

How can we find the product of variable terms that have numerical coefficients other than 1? First, we must use the commutative and associative properties of multiplication to rearrange and group factors so that numerical factors are grouped together and variable factors with like bases are grouped together. Then, we can multiply the numerical factors and use the Product of Powers Property to multiply variable factors with like bases.

EXAMPLE: $(3a^2b^2)(-2a^2bc^3)$

Group the numerical factors 3 and -2 together, the variable factors a^2 and a^2 together, and the variable factors b^2 and b together.

$$= \underbrace{(3 \times -2)}_{\text{NUMERICAL FACTORS}}\underbrace{(a^2 \times a^2)}_{\text{FACTORS WITH BASE `a'}}\underbrace{(b^2 \times b)}_{\text{FACTORS WITH BASE `b'}}c^3$$

Now multiply the numerical factors 3 and -2 and use the Product of Powers Property to find the products $(a^2 \times a^2)$ and $(b^2 \times b)$.

$$= -6a^{2+2}b^{2+1}c^3$$

$$= \underline{\underline{-6a^4b^3c^3}}$$

Self-correcting Exercise #11

Answers may be found on page 59 of this lesson.

1. Find the following products.

(a) $x^5 \times x^2 =$

(b) $m^{-4} \times m^2 =$

(c) $2y^3 \times -5y =$

(d) $(-ab)(3a^2) =$

(e) $(-7x^2y)(-2xy^2) =$

(f) $(6y^2)(-4y^2) =$

(g) $(\frac{1}{2}xy)(-2xy^3) =$

(h) $(6cd)(-5c^2)(3d^2) =$

(i) $(-4m^2n^3)(-mn^4)(-2m^3n) =$

D. Dividing Variable Terms

The Quotient of Powers Property tells us that in order to divide two powers with like bases, we keep the common base and subtract the exponents. That is, for any integral powers a^m and a^n ,

$$\frac{a^m}{a^n} = a^{m-n}$$

Since all variables represent real numbers, this property can also be applied when the base of a power is a variable rather than a real number.

EXAMPLES:

$$\frac{y^5}{y^4} = y^{5-4} = y$$

$$\frac{c^3}{c} = c^{3-1} = c^2$$

$$\frac{x^2}{x^4} = x^{2-4} = x^{-2} \text{ (or) } \frac{1}{x^2}$$

Note that the last example could have been done by subtracting the exponents in the reverse manner.

i.e. $\frac{x^2}{x^4} = \frac{1}{x^{4-2}} = \frac{1}{x^2}$

In dividing powers with variable bases, we usually subtract the exponents in the manner that will give the result a positive exponent.

EXAMPLES:

$$\frac{m^3}{m^4} = \frac{1}{m^{4-3}} = \frac{1}{m}$$

$$\frac{m^4}{m^3} = m^{4-3} = m$$

How can we find the quotient of variable terms that have numerical coefficients other than 1? We must think of the quotient as being a product so that the commutative and associative properties of multiplication can be used to group numerical factors together and literal factors with like bases together. (Remember that multiplication is commutative and associative while division is not.) Then, we can divide the numerical factors and use the Quotient of Powers Property to divide variable factors with like bases.

EXAMPLE:

$$\frac{-4m^2n}{2mn^4}$$

Think of the quotient as being a product.

$$= -4m^2n \times \frac{1}{2mn^4} \longleftarrow \text{THINK THIS STEP DON'T WRITE IT.}$$

The commutative and associative properties of multiplication can now be used to group numerical factors and group literal factors with like bases.

$$= \frac{-4}{2} \times \frac{m^2}{m} \times \frac{n}{n^4}$$

$$= -2 \times m \times \frac{1}{n^3}$$

$$= \boxed{\frac{-2m}{n^3}}$$

Self-correcting Exercise #12

Answers may be found on page 59 of this lesson.

1. Simplify the following quotients. (Express all answers with positive exponents.)

(a) $\frac{y^4}{y^2} =$ (b) $\frac{x}{x^3} =$

(c) $\frac{m^9}{m} =$ (d) $\frac{r^2}{r^5} =$

(e) $\frac{24y^4}{-3y^3} =$ (f) $\frac{-6b}{12b^2} =$

(g) $\frac{-15a^3b^2c^2}{12ab} = \frac{-15}{12} \times \frac{a^3}{a} \times \frac{b^2}{b} \times c^2 =$

(h) $\frac{4ab^2c^2}{2b^4c} =$

(i) $\frac{6m^5n^2}{-2m^3n^3} =$

EXERCISE - Operating With Variable Terms

1. Fill in the blanks.

(a) A ______________ is a symbol which may represent any of the elements of a specified set.

(b) The expression $x^2 + 1$ instructs us to ______________ a number and then ______________ the result by ________.

(c) The expression -2(y - 5) instructs us to ______________ a number by _____ and then ______________ the result by _____.

(d) The set of values that may be substituted for a variable is called the ______________ of the variable.

(e) In the algebraic term -5ab, the three ______________ involved are -5, a, and b. The numerical coefficient of this term is _____ and the literal coefficient is _____.

(f) The ________________ of ________________ property tells us that $y^2 \times y = y^3$.

(g) The ________________ of ________________ property tells us that $\frac{m^4}{m} = m^3$.

(h) The algebraic terms $5m^2n$ and $-3m^2n$ are ______________ terms because they have the same ______________________ coefficient.

(i) In order to add like terms, we add the ________________ coefficients and multiply this sum by the common ________________ coefficient.

2. Simplify by combining like terms.

(a) $3m + 5m =$ __________ (b) $6y - 8y =$ __________

(c) $5a^2d - a^2d =$ __________ (d) $5pq^2r - 5pq^2r =$ __________

(e) $-4y^2 + 6y^2 =$ __________ (f) $-8w - 10w =$ __________

(g) $-12mn + 7mn =$ __________ (h) $-6s + 6s =$ __________

(i) $4b - 6b + 5a - a$

$=$

(j) $p^2 + 2pq - q^2 + 4p^2 + 3pq - 2q^2$

$=$

(k) $-3x - 2y + 5x + y - 3x$

$=$

(l) $\sqrt{5}x + 4\sqrt{5}x - 2\sqrt{5}x$ (notice that x is not under the root symbol)

$=$

(m) $a^2 - 2ab + b^2 + 3ab - 4a^2 + 6b^2 - 12ab$

$=$

(n) $x + y + z - 3x - 2y + 4z - 2x + y - 3z$

$=$

(o) $2m^2 - 3mn + 4n^2 - 3n^2 - 2mn + n^2 - m^2$

$=$

(p) $ab - ac + b^2 - 4ac + c^2 + 3bc - 5ab + 7bc - 4c^2$

$=$

3. Multiply.

(a) $y^3 \times y^3 =$

(b) $-x^2 \times x^4 =$

(c) $-b \times -b^3 =$

(d) $2x^2 \times x =$

(e) $-3ab \times 4b^2 =$

(f) $\sqrt{2}x \times \sqrt{2}x =$
*(see page 18)

(g) $-7xy^2 \times -2x^2y$
$= (-7 \times -2)(x \times x^2)(y^2 \times y)$
$=$

(h) $4abc \times -b^2c^4$
$=$

(i) $8rst \times 3r^2t^3$
$=$

(j) $5x^5y^3 \times 2x^2y^2$
$=$

4. Divide. (Express all answers with positive exponents.)

(a) $\frac{y^8}{y^2} =$

(b) $\frac{x}{x^5} =$

(c) $\frac{-m^2n}{m} =$

(d) $\frac{-x^2y}{-x^3y^2} =$

(e) $\frac{32abc}{8abc} =$

(f) $\frac{-6x^3y^2}{2x^4y} =$

(g) $\frac{4abc}{-2b^2} =$

(h) $\frac{18pq}{-9q^2} =$

(i) $\frac{-12a^2b}{-4ab} =$

(j) $\frac{36b^5c^4}{24b^2c} =$

Topic Four: Powers and Roots of Variable Terms

A. Some More Power Properties

You have already been told that the Product of Powers Property and the Quotient of Powers Property can be used to find the product or quotient of powers with like variable bases. For example, according to these properties:

$$y^2 \times y^3 = y^5$$
$$\frac{x^4}{x^2} = x^2$$

We need some additional power properties to help us work with powers with variable bases. These properties are discussed in the sections that follow.

I. Power of a Product Property

When the exponent of a power is a positive integer, it tells us how many times the base of the power is to be used as a factor. For example, the power $(-4xy)^3$ tells that the base $-4xy$ is to be used as a factor 3 times.

i.e. $(-4xy)^3 = -4xy \times -4xy \times -4xy$
$= -64x^3y^3$

This same result could have been obtained by applying the exponent 3 to each factor in the product $-4xy$.

i.e. $(-4xy)^3 = (-4)^3x^3y^3$
$= -64x^3y^3$

In general, any power with a variable base can be simplified by applying the exponent to every factor in the base.

POWER OF A PRODUCT PROPERTY

For any positive integer m ,

$$(a \times b \times c \times \ldots)^m = a^m \times b^m \times c^m \times \ldots$$

EXAMPLES:

$$(3mn)^4 = 3^4 m^4 n^4 = 81 m^4 n^4$$
$$(-2y)^5 = (-2)^5 y^5 = -32y^5$$

Use the Power of a Product Property to simplify the following powers.

1. $(9x)^2 = \underline{\quad}^2 \, \underline{\quad}^2 = \underline{\qquad}$
2. $(-3abc)^3 = (\underline{\quad})^3 \, \underline{\quad}^3 \, \underline{\quad}^3 \, \underline{\quad}^3 = \underline{\qquad\quad}$
3. $(-5x)^4 =$
4. $(3m)^5 =$

II. Power of a Quotient Property

Some powers have variable bases that are fractional in form. For example, consider the power $\left(\frac{x}{y}\right)^4$. It tells us that the base $\frac{x}{y}$ is to be used as a factor 4 times.

i.e. $$\left(\frac{x}{y}\right)^4 = \frac{x}{y} \times \frac{x}{y} \times \frac{x}{y} \times \frac{x}{y}$$

$$= \frac{x^4}{y^4}$$

The same result could have been obtained by applying the exponent 4 to the numerator and denominator of the fraction $\frac{x}{y}$.

i.e. $$\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$$

In general, any power with a fractional base can be simplified by applying the exponent to both terms of the fraction.

POWER OF A QUOTIENT PROPERTY

For any positive integer m,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

EXAMPLES:

$$\left(\frac{y}{z}\right)^3 = \frac{y^3}{z^3}$$

$$\left(\frac{t}{u}\right)^5 = \frac{t^5}{u^5}$$

Use the Power of a Quotient Property to simplify the following powers.

1. $\left(\frac{-4}{b}\right)^2 = \frac{(\underline{\quad})^2}{\underline{\quad}^2} = \underline{\qquad}$

2. $\left(\frac{m}{n}\right)^7 =$

3. $\left(\frac{-y}{x}\right)^4 =$

4. $\left(\frac{1}{a}\right)^3 =$

III. Power of a Power Property

Some powers have bases that are also powers. For example, consider the power $(x^2)^3$. It tells us that the base x^2 is to be used as a factor 3 times.

$$\text{i.e.}\quad \begin{aligned}(x^2)^3 &= x^2 \times x^2 \times x^2\\ &= x^{2+2+2}\\ &= x^6\end{aligned}$$

The same result could have been obtained by retaining the same base x and multiplying the exponents.

$$\text{i.e.}\quad \begin{aligned}(x^2)^3 &= x^{2\times 3}\\ &= x^6\end{aligned}$$

In general, a power of a power can be written with a single exponent by retaining the same base and multiplying the exponents.

POWER OF A POWER PROPERTY

For all positive integers m and n,

$$(a^m)^n = a^{mn}$$

EXAMPLES:

$$(y^3)^4 = y^{3\times4} = y^{12}$$

$$(m^2)^2 = m^{2\times2} = m^4$$

Use the Power of a Power Property to simplify the following powers.

1. $(r^3)^5 = r^{3\times5} = r$—
2. $(y^3)^2 =$
3. $(m^2)^4 =$
4. $(m^4)^3 =$

In simplifying some powers with variables bases, you must use all three of the Power Properties discussed on pages 39 - 42.

EXAMPLE:

$$\left(\frac{-2x^2}{3y^3}\right)^3$$

First, you must use the Power of a Quotient Property so that the exponent 3 can be applied to both the numerator and denominator of the fractional base.

$$= \frac{(-2x^2)^3}{(3y^3)^3}$$

Next, you must use the Power of a Product Property so that the exponent 3 can be applied to all the factors in the numerator and denominator.

$$= \frac{(-2)^3(x^2)^3}{3^3(y^3)^3}$$

Finally, you must use the Power of a Power Property to write the factors $(x^2)^3$ and $(y^3)^3$ with a single exponent.

$$= \boxed{\frac{-8x^6}{27y^9}}$$

Self-correcting Exercise #13

Answers may be found on page 60 of this lesson.

1. Write the following powers in simplest form.

   (a) $(-7ab)^2 =$

   (b) $(3y)^3 =$

   (c) $\left(\frac{-6}{x}\right)^2 =$

   (d) $\left(\frac{2ab}{3c}\right)^4 =$

   (e) $\left(\frac{-2m^3}{5n^2}\right)^2 =$

B. Zero and Negative Integral Powers of Variable Terms

On page 9 of this lesson, we defined the zero power of any non-zero real number base to be equal to 1. That is, for any non-zero real number a,

$$a^0 = 1 .$$

Since all variables represent real numbers, this definition also applies to powers with variable bases.

EXAMPLES:

$$y^0 = 1, \quad (-3x)^0 = 1, \quad (abc)^0 = 1, \quad \left(\frac{5xy}{z}\right)^0 = 1$$

On page 10 of this lesson, we defined the negative integral power a^{-m} to be the reciprocal of the positive integral power a^m. That is, for any negative integral power a^{-m},

$$a^{-m} = \frac{1}{a^m}$$

Negative integral powers with variable bases can be defined in a similar manner.

EXAMPLES:

$$y^{-3} = \frac{1}{y^3}$$

$$\frac{1}{m^{-2}} = \frac{1}{\frac{1}{m^2}} = 1 \div \frac{1}{m^2} = 1 \times \frac{m^2}{1} = m^2$$

$$(2y)^{-5} = \frac{1}{(2y)^5} = \frac{1}{2^5 y^5} = \frac{1}{32y^5}$$

$$\frac{3x^{-5}}{2y} = \frac{3 \times \frac{1}{x^5}}{2y} = \frac{3}{x^5} \div 2y = \frac{3}{x^5} \times \frac{1}{2y} = \frac{3}{2x^5 y}.$$

All the properties that we have used for working with positive integral powers also apply to negative integral powers.

EXAMPLES:

$y^3 \times y^{-1} = y^{3+(-1)} = y^2$ (Product of Powers Property)

$\frac{x^2}{x^{-3}} = x^{2-(-3)} = x^5$ (Quotient of Powers Property)

$(ab)^{-3} = a^{-3}b^{-3} \left(\text{or } \frac{1}{a^3b^3}\right)$ (Power of a Product Property)

$\left(\frac{x}{y}\right)^{-1} = \frac{x^{-1}}{y^{-1}} \left(\text{or } \frac{y}{x}\right)$ (Power of a Quotient Property)

$(x^2)^{-3} = x^{2 \times -3} = x^{-6} \left(\text{or } \frac{1}{x^6}\right)$ (Power of a Power Property)

Self-correcting Exercise #14

Answers may be found on page 60 of this lesson.

1. Write the following expressions with positive exponents.

(a) $3y^{-5} =$

(b) $b^{-2} =$

(c) $\frac{1}{x^{-3}} =$

(d) $\frac{1}{4y^{-7}} =$

(e) $\frac{3ab^{-1}}{c^{-1}} =$

2. Simplify the following expressions.

(a) $(2y)^{-3} =$

(b) $\frac{x^{-2} \times x^{5}}{x^{2}} =$

(c) $\frac{(ab)^{2}}{b^{-3} \times a^{2}} =$

(d) $\frac{(a^{2})^{-4} \times (a^{5})^{3}}{(a^{-2})^{-2}} =$

(e) $\left(\frac{3a^{2}}{4b}\right)^{-3} =$

C. Roots of Variable Terms

The square root of a variable term represents a real number only when the variable term represents a non-negative real number. For example, if we take the principal square root of the variable term x, we arrive at the variable expression $\sqrt{x}$. The radical $\sqrt{x}$ represents a real number only when $x \geqq 0$. Note that:

When $x = 4$, $\sqrt{x} = \sqrt{4} = 2$ which is a real number.
When $x = -4$, $\sqrt{x} = \sqrt{-4}$ which is not a real number.
When $x = 7$, $\sqrt{x} = \sqrt{7}$ which is a real number.
When $x = -7$, $\sqrt{x} = \sqrt{-7}$ which is not a real number.

Evaluate each radical below for the values of the variables that are stated. Then decide if the radical represents a real number when the variables assume these values.

Radical	Values of Variables	Substitution (Give answer in simplest form.)	Is Radical Now a Real Number?
$\sqrt{3ab}$	$a = 1, \; b = 5$	$\sqrt{3(1)(5)} = \sqrt{15}$	yes
$\sqrt{y}$	$y = -2$		
$\sqrt{x + 1}$	$x = 0$		
$\sqrt{x^2}$	$x = -2$		
$\sqrt{x^3}$	$x = -2$		
$\sqrt{x^4}$	$x = -1$		
$\sqrt{a^2 b}$	$a = -1, \; b = 2$		
$\sqrt{ab^2}$	$a = -1, \; b = 2$		
$\sqrt{xy}$	$x = -2, \; y = 3$		
$\sqrt{-2ab}$	$a = -3, \; b = 2$		

Now let us consider the principal square root of the variable term x^2. The principal square root can be represented by the symbol $\sqrt{x^2}$. At first glance, it appears that $\sqrt{x^2}$ should equal x. But, remember that a principal square root must always be positive. Thus, $\sqrt{x^2}$ equals x if x represents a non-negative real number but it does not equal x if x represents a negative number. Therefore, we must find some way to represent $\sqrt{x^2}$ so that the result will always be positive or zero. To do this, we use the absolute value symbol which was discussed in Lesson B. If we take the absolute value of x, written $|x|$, we know that this value is always non-negative. Thus, we must define the principal square root of x^2, $\sqrt{x^2}$, to be equal to the absolute value of x, $|x|$.

> For any variable x whose domain is R,
>
> $$\sqrt{x^2} = |x|.$$

The above rule guarantees that the principal square root of a variable term will always be positive or zero, regardless of the value substituted for the variable. For example,

if x = 5, $\sqrt{x^2} = \sqrt{(5)^2} = |5| = 5$

if x = -5, $\sqrt{x^2} = \sqrt{(-5)^2} = |-5| = 5$

Both results are positive.

Any square root whose radicand is a variable term that is a perfect square can be written without the radical sign. An absolute value sign must be placed around the result. Then, any factors that are always positive can be removed from the absolute value sign. (Any power with a variable base and an even exponent is always positive.)

EXAMPLES:

$$\sqrt{y^2} = |y|$$

$$\sqrt{(x+1)^2} = |x+1|$$

$$\sqrt{9x^2} = \sqrt{(3x)^2} = |3x| = 3|x|$$

The factor 3 can be taken outside the absolute value sign since 3 is always positive.

$$\sqrt{x^4} = \sqrt{(x^2)^2} = |x^2| = x^2$$

Absolute value sign can be dropped here since x^2 is always positive.

$$\sqrt{y^6} = \sqrt{(y^3)^2} = |y^3|$$

Absolute value sign must be retained since y^3 can be positive or negative.

Write the following radicals without the radical sign.

1. $\sqrt{a^2b^2} = \sqrt{(\_\_\_\_\_)^2} = |\_\_\_|$
2. $\sqrt{25y^2} = \sqrt{(\_\_\_\_\_)^2} = |\_\_\_| = \_\_\_|\_\_\_|$
3. $\sqrt{16a^2b^4c^6} = \sqrt{(\_\_\_\_\_)^2} = |\_\_\_\_\_| = \_\_\_\_\_|\_\_\_|$

EXERCISE - Powers and Roots of Variable Terms

1. Fill in the blanks.

(a) x^{-4} is called a ____________ integral power since its ____________ is a ____________________ integer. By definition, x^{-4} is equal to the ____________________ of x^4.

(b) The ____________ of a ____________ property tells us that $(x^3)^4 = x^{12}$.

(c) In order to simplify the power $\left(\frac{2xy}{z}\right)^3$, we must use the ____________ of a ____________ property and the ____________ of a ____________ property.

(d) The Power of a Product Property tells us that the ______________ of a power can be applied to every ______________ in the base.

(e) The radical $\sqrt{y}$ represents a real number only when y is a ____________ real number or zero.

(f) By definition, the principal square root of y^2 is equal to the ____________ value of y.

2. Simplify the following radicals. (Remember that any factors that are always positive can be removed from the absolute value sign.)

(a) $\sqrt{x^2} =$

(b) $\sqrt{m^2n^2} = \sqrt{(mn)^2} =$

(c) $\sqrt{9y^2} = \sqrt{(\quad)^2} = |\quad\quad| = \_\_\_ \, |y|$

(d) $\sqrt{100z^2} =$

(e) $\sqrt{s^6} =$

(f) $\sqrt{y^4} =$

(g) $\sqrt{49x^{10}} =$

(h) $\sqrt{4x^2y^6} = \sqrt{(\quad)^2} =$

3. Name the property or definition which justifies each statement. Choose from the following list.

- product of powers property
- quotient of powers property
- power of a product property
- power of a quotient property
- power of a power property
- definition of zero exponent
- definition of negative exponents

(a) $y^3 \times y^3 = y^6$ product of powers property

(b) $(y^2)^5 = y^{10}$ ____________________

(c) $(xy)^0 = 1$ ____________________

(d) $(3ab)^3 = 27a^3b^3$ ____________________

(e) $\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$ ____________________

(f) $2mn \times 3n^3 = 6mn^4$ ____________________

(g) $\frac{1}{y^{-2}} = y^2$ ____________________

(h) $(ab)^{-2} = a^{-2}b^{-2}$ ____________________

(i) $\frac{4y^2}{y} = 4y$ ____________________

(j) $x^0 \times y^4 = y^4$ ____________________

(k) $3m^{-2} = \frac{3}{m^2}$ ____________________

(l) $\left(\frac{2ab}{d}\right)^3 = \frac{(2ab)^3}{d^3}$ ____________________

4. Each of the statements below is false. Alter the right side of each equation in order to make it a true statement.

(a) $xy^{-5} = \frac{1}{xy^5}$

$xy^{-5} = \frac{x}{y^5}$

(b) $x^3 \times x^2 = x^6$

(c) $(m^3)^5 = m^8$

(d) $\left(\frac{a}{b}\right)^{-2} = \frac{a^2}{b^2}$

(e) $(3b)^2 = 3b^2$

(f) $x^{-1} + y^{-1} = \frac{1}{x+y}$

(g) $\frac{x^4}{x^4} = 0$

(h) $\frac{1}{5x^{-3}} = 5x^3$

(i) $(2x)^{-5} = \frac{1}{2x^5}$

(j) $(ab)^2 = a \times b^2$

(k) $(5y)^0 = 0$

(l) $\left(\frac{2x}{y}\right)^3 = \frac{2x^3}{y^3}$

5. Simplify the following powers.

(a) $(2b)^5 =$ $2^5 \times b^5 = 32b^5$

(b) $(-3y)^3 =$

(c) $(7m)^2 =$

(d) $(4a^2b)^3 =$

(e) $\left(\frac{x}{y}\right)^{-1} =$

(f) $(3x)^{-2} =$

(g) $\left(\frac{3m^2}{n}\right)^4 =$

(h) $\left(\frac{-2x}{y}\right)^6 =$

(i) $\left(\frac{8a^3}{bc}\right)^2 =$

(j) $\left(\frac{5a}{b}\right)^{-2} =$

6. Use the power properties to simplify the following expressions. (Review the second set of examples on p.44.)

(a) $3x^{-2} \times 4x^4 = (3 \times 4)\ (x^{-2} \times x^4) = 12x^{-2+4} =$ ____________

(b) $\frac{8y^3}{-2y^{-1}} =$

(c) $-3y^{-3} \times 5y^{-1} =$

(d) $\frac{(xy)^3}{x^2 \times y} =$

(e) $(-2ab)^2 - 3a^2b^2 =$

(f) $\frac{(x^{-5})^{-3}}{(x^{-2})^2} = \frac{x^{-5 \times -3}}{x^{-2 \times 2}} =$

(g) $\left(\frac{2a}{b}\right)^3 \times \frac{a^2}{4b} =$

(h) $\frac{(-2x^2y)^3}{x^3 \times x^{-1}} =$

7. Write each expression with positive exponents only.

(a) $5ab^{-1} =$

(b) $2c^{-5}d^{-2} =$

(c) $\frac{1}{8x^{-5}} = \frac{1}{8 \times \frac{1}{x^5}} =$

(d) $\frac{1}{3xy^{-2}} =$

(e) $\frac{4}{x^{-3}} =$

8. The domain of each variable is R. First, choose a value for each variable that will make the expression represent a real number. Then choose a value for each variable so that the expression does not represent a real number.

EXAMPLE: $\sqrt{-2ab}$

(i) When $a = -1$ and $b = 3$,

$\sqrt{-2ab} = \sqrt{-2(-1)(3)} = \sqrt{6}$ which is a real number.

(ii) When $a = 1$ and $b = 3$,

$\sqrt{-2ab} = \sqrt{-2(1)(3)} = \sqrt{-6}$ which is not a real number.

(a) $\sqrt{x - 3}$

(i) ______________________________

(ii) ______________________________

(b) $\sqrt{y^3}$

(i) ______________________________

(ii) ______________________________

(c) $\sqrt{2x + 1}$

(i) ______________________________

(ii) ______________________________

(d) $\sqrt{3x^2}$

(i) ______________________________

(ii) ______________________________

Key to Self-correcting Exercises in Lesson C

Exercise #1, page 3

1.

Power	Base	Exponent	Evaluation
$(-6)^3$	-6	3	$-6 \times -6 \times -6 = -216$
$\left(\frac{5}{8}\right)^2$	$\frac{5}{8}$	2	$\frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$
$(0.2)^4$	0.2	4	$0.2 \times 0.2 \times 0.2 \times 0.2 = 0.0016$
$(-4)^4$	-4	4	$-4 \times -4 \times -4 \times -4 = 256$
ALWAYS USE PARENTHESES AROUND NEGATIVE BASES.			
$\left(\frac{-1}{3}\right)^5$	$\frac{-1}{3}$	5	$\frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} \times \frac{-1}{3} = \frac{-1}{243}$

2. (a) $100{,}000 = 10^5$ (b) $125 = 5^3$ (c) $\frac{4}{9} = \left(\frac{2}{3}\right)^2$

(d) $121 = 11^2$ (e) $\frac{27}{1000} = \left(\frac{3}{10}\right)^3$ (f) $128 = 2^7$

3. (a) $36 = (-6)^2$ (b) $-216 = (-6)^3$ (c) $-32 = (-2)^5$

(d) $100 = (-10)^2$ (e) $\frac{-27}{64} = \left(\frac{-3}{4}\right)^3$ (f) $144 = (-12)^2$

4. (a) negative (since 7 is odd) (b) positive (since 16 is even) (c) negative (since 9 is odd)

(d) negative
$-8^4 = -(8 \times 8 \times 8 \times 8)$
$= -4096$
(e) positive (since 4 is even)
(f) positive [$(-5)^{31}$ is negative, so $-(-5)^{31}$ is positive.]

5. (a) $(-8)^3 + 5^4 = (-8 \times -8 \times -8) + (5 \times 5 \times 5 \times 5) = -512 + 625 = \boxed{113}$

(b) $-3(4^3 - 5^2) = -3[(4 \times 4 \times 4) - (5 \times 5)] = -3(64 - 25) = -3(39) = \boxed{-117}$

(c) $\left(\frac{-2}{3}\right)^3 - \frac{5}{9} = \left(\frac{-2}{3} \times \frac{-2}{3} \times \frac{-2}{3}\right) - \frac{5}{9} = \frac{-8}{27} - \frac{5}{9} = \frac{-8 \;-15}{27} = \boxed{\frac{-23}{27}}$

(d) $\frac{-3 + (-2)^5}{7} = \frac{-3 + (-2 \times -2 \times -2 \times -2 \times -2)}{7} = \frac{-3 + (-32)}{7} = \frac{-35}{7} = \boxed{-5}$

(e) $4 - (-3)^2 = 4 - (-3 \times -3) = 4 - 9 = \boxed{-5}$

Exercise #2, page 7

1. (a) $7^3 \times 7^4 = 7^{3+4} = \boxed{7^7}$

(b) $(-3)(-3)(-3)^6 = (-3)^{1+1+6} = (-3)^8$

(c) $\frac{5^9}{5} = 5^{9-1} = \boxed{5^8}$

(d) $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^{2+3+1+4} = \left(\frac{1}{2}\right)^{10}$

(e) $\frac{(-6)^4 \; (-6)^8}{(-6)^7} = \frac{(-6)^{4+8}}{(-6)^7} = \frac{(-6)^{12}}{(-6)^7} = (-6)^{12-7} = (-6)^5$

(f) $\frac{4^{15}}{4^3 \times 4^5} = \frac{4^{15}}{4^{3+5}} = \frac{4^{15}}{4^8} = 4^{15-8} = \boxed{4^7}$

2. (a) $\frac{(-5)^{20}(-5)^3}{(-5)^{21}} = (-5)^{20+3-21} = (-5)^2 = \boxed{25}$

(b) $\frac{3^{11} \times 3^5 \times 2^{13}}{3^{14} \times 2^9} = 3^{11+5-14} \times 2^{13-9} = 3^2 \times 2^4 = 9 \times 16 = \boxed{144}$

(c) $\frac{6^8}{6^5} - 6^2 = (6^{8-5} - 6^2) = (6^3 - 6^2) = 216 - 36 = \boxed{180}$

Exercise #3, page 11

1. (a) $243 = 3^5$

(b) $1 = 3^0$

(c) $\frac{1}{27} = 3^{-3}$

(d) $\frac{1}{9} = 3^{-2}$

(e) $9 = 3^2$

(f) $\frac{1}{81} = 3^{-4}$

2. (a) $4^0 = 1$

(b) $(-1)^3 = -1 \times -1 \times -1 = -1$

(c) $4^{-3} = \frac{1}{4^3} = \frac{1}{4 \times 4 \times 4} = \frac{1}{64}$

(d) $(-5)^{-2} = \frac{1}{(-5)^2} = \frac{1}{-5 \times -5} = \frac{1}{25}$

(e) $(0.7)^2 = 0.7 \times 0.7 = 0.49$

(f) $\left(\frac{1}{2}\right)^{-4} = \frac{1}{\left(\frac{1}{2}\right)^4} = \frac{1}{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}} = \frac{1}{\frac{1}{16}} = 16$

Exercise #3, (#2, continued)

(g) $4 \times 3^{-1} = 4 \times \frac{1}{3} = \frac{4}{3}$

(h) $2^{-3} \times 4^2 = \frac{1}{2^3} \times 4^2 = \frac{1}{8} \times 16 = 2$

(i) $\left(\frac{3}{4}\right)^{-3} \times 9^2 = \frac{1}{\left(\frac{3}{4}\right)^3} \times 81 = \frac{1}{\frac{27}{64}} \times 81 = \frac{64}{\underset{1}{\cancel{27}}} \times \overset{3}{\cancel{81}} = 192$

(j) $\frac{6}{8^{-2}} = \frac{6}{\frac{1}{8^2}} = 6 \times \frac{8^2}{1} = 6 \times 64 = 384$

3. (a) $6^{-8} = \frac{1}{6^8}$

(b) $\frac{1}{4^{-6}} = \frac{1}{\frac{1}{4^6}} = 4^6$

(c) $3 \times 5^{-7} = 3 \times \frac{1}{5^7} = \frac{3}{5^7}$

(d) $\frac{1}{2^{-9}} + 3^{-5} = 2^9 + \frac{1}{3^5}$

(e) $\frac{7^{-5}}{4^{-8}} = \frac{4^8}{7^5}$

4. (a) $6^{18} \times 6^{-6} = 6^{18-6} = 6^{12}$

(b) $\frac{4^8}{4^{-2}} = 4^{8-(-2)} = 4^{8+2} = 4^{10}$

(c) $\frac{3^{-9}}{3^{-3}} = 3^{-9-(-3)} = 3^{-9+3} = 3^{-6} \left(\text{or } \frac{1}{3^6}\right)$

(d) $4^{-5} \times 4^{-7} \times 4^3 = 4^{-5-7+3} = 4^{-9} \left(\text{or } \frac{1}{4^9}\right)$

(e) $\frac{2^{-4} \times 2^{-3}}{2^5} = \frac{2^{-7}}{2^5} = 2^{-7-5} = 2^{-12} \left(\text{or } \frac{1}{2^{12}}\right)$

Exercise #4, page 17

1. (a) $\sqrt{144} = 12$

(b) $-\sqrt{144} = -12$

(c) $(\sqrt{0.9})^2 = 0.9$

(d) $\sqrt{\frac{4}{25}} = \frac{2}{5}$

(e) $\sqrt{6} \times \sqrt{6} = 6$

(f) $-\sqrt{3} \times -\sqrt{3} = 3$

(g) $-\sqrt{81} = -9$

(h) $\sqrt{3600} = 60$

2. (a) irrational (17 is not a perfect square.)

(b) rational $(-\sqrt{36} = -6)$

(c) imaginary (Square roots of negative numbers are imaginary.)

(d) rational $\sqrt{\frac{81}{100}} = \frac{9}{10}$

(e) irrational $\left(\frac{2}{5}\right.$ is not a perfect square.$\left.\right)$

(f) imaginary (Square roots of negative numbers are imaginary.)

Exercise #5, page 19

1. (a) $\sqrt{5} \times \sqrt{7} = \sqrt{35}$

(b) $-\sqrt{3} \times \sqrt{7} = -\sqrt{21}$

(c) $\sqrt{13} \times \sqrt{13} = 13$

(d) $-2\sqrt{2} \times -\sqrt{3} = 2\sqrt{6}$

(e) $(\sqrt{5})^2 = 5$

(f) $-2\sqrt{3} \times 3\sqrt{5} = -6\sqrt{15}$

(g) $\sqrt{11} \times \sqrt{5} \times \sqrt{2} = \sqrt{110}$

(h) $3\sqrt{2} \times -2\sqrt{5} \times 5\sqrt{3} = -30\sqrt{30}$

(i) $(-\sqrt{10})^2 = 10$

(j) $5\sqrt{\frac{2}{3}} \times 3\sqrt{\frac{3}{5}} = 15\sqrt{\frac{2}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{3}}}{5}} = 15\sqrt{\frac{2}{5}}$

Exercise #6, page 21

1. (a) $\frac{2\sqrt{7}}{\sqrt{3}} = \frac{2\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{21}}{3}$

(b) $\frac{-27\sqrt{6}}{9\sqrt{3}} = \frac{-27}{9} \times \sqrt{\frac{6}{3}} = -3\sqrt{2}$

(c) $\frac{-8\sqrt{42}}{-2\sqrt{7}} = \frac{-8}{-2} \times \sqrt{\frac{42}{7}} = 4\sqrt{6}$

(d) $\frac{-2}{\sqrt{15}} = \frac{-2}{\sqrt{15}} \times \frac{\sqrt{15}}{\sqrt{15}} = \frac{-2\sqrt{15}}{15}$

Exercise #7, page 22

1. (a) $\boxed{\sqrt{2},}$ $\boxed{3\sqrt{2},}$ $3\sqrt{5},$ $2\sqrt{3}$
Circled terms have a radicand of 2.

(b) $\boxed{\sqrt{11},}$ $11\sqrt{3},$ $\boxed{2\sqrt{11},}$ $\boxed{-\sqrt{11}}$
Circled terms have a radicand of 11.

(c) $6\sqrt{5},$ $\boxed{5\sqrt{7},}$ $\boxed{3\sqrt{7},}$ $5\sqrt{6}$
Circled terms have a radicand of 7.

(d) $2\sqrt{10},$ $\boxed{7\sqrt{6},}$ $10\sqrt{2},$ $\boxed{-\sqrt{6},}$ $\sqrt{-6}$
Circled terms have a radicand of 6.

2. (a) $5\sqrt{6} + 9\sqrt{6} = (5 + 9)\sqrt{6} = 14\sqrt{6}$

(b) $-2\sqrt{3} + 7\sqrt{3} = (-2 + 7)\sqrt{3} = 5\sqrt{3}$

(c) $-6\sqrt{5} - 8\sqrt{5} = (-6 - 8)\sqrt{5} = -14\sqrt{5}$

(d) $4\sqrt{7} - \sqrt{7} = (4 - 1)\sqrt{7} = 3\sqrt{7}$

(e) $3\sqrt{2} + 7\sqrt{2} - 12\sqrt{2}$
$= (3 + 7 - 12)\sqrt{2}$
$= -2\sqrt{2}$

(f) $2\sqrt{5} + 3\sqrt{2} - \sqrt{5} - 6\sqrt{2}$
$= (2 - 1)\sqrt{5} + (3 - 6)\sqrt{2}$
$= \sqrt{5} - 3\sqrt{2}$

Exercise #8, page 24

PERFECT SQUARE FACTOR

1. (a) $\sqrt{32} = \sqrt{\boxed{16} \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$

(b) $\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$

(c) $\sqrt{700} = \sqrt{100 \times 7} = \sqrt{100} \times \sqrt{7} = 10\sqrt{7}$

(d) $\sqrt{243} = \sqrt{81 \times 3} = \sqrt{81} \times \sqrt{3} = 9\sqrt{3}$

SAME AS MULTIPLYING BY 1.

2. (a) $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} \times \boxed{\frac{\sqrt{2}}{\sqrt{2}}} = \frac{\sqrt{2}}{2}$

(b) $\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$

(c) $-3\sqrt{\frac{2}{5}} = \frac{-3\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{-3\sqrt{10}}{5}$

PERFECT SQUARE

3. (a) $3\sqrt{2} \times 5\sqrt{10} = 15\sqrt{20} = 15\sqrt{4 \times 5} = 15 \times \sqrt{4} \times \sqrt{5} = 15 \times 2 \times \sqrt{5} = 30\sqrt{5}$

(b) $3\sqrt{28} - 5\sqrt{63} = 3\sqrt{4 \times 7} - 5\sqrt{9 \times 7} = (3 \times 2\sqrt{7}) - (5 \times 3\sqrt{7}) = 6\sqrt{7} - 15\sqrt{7} = -9\sqrt{7}$

(c) $\frac{5\sqrt{6}}{3\sqrt{2}} = \frac{5}{3} \times \sqrt{\frac{6}{2}} = \frac{5}{3}\sqrt{3}$

(d) $2\sqrt{75} - 3\sqrt{108} + \sqrt{147} = 2\sqrt{25 \times 3} - 3\sqrt{36 \times 3} + \sqrt{49 \times 3}$
$= (2 \times 5\sqrt{3}) - (3 \times 6\sqrt{3}) + 7\sqrt{3}$
$= 10\sqrt{3} - 18\sqrt{3} + 7\sqrt{3}$
$= -\sqrt{3}$

Exercise #9, page 29

1. (a) multiply, add

(b) square, divide, 2

(c) subtract, multiply, $\frac{2}{3}$

(d) multiply, 3, principal

(e) multiply, 2, square

2. (a) $-5y^3 = -5(-2)^3 = -5(-2 \times -2 \times -2) = -5(-8) = \textcircled{40}$

(b) $3(x + 1)^2 = 3(-5 + 1)^2 = 3(-4)^2 = 3(-4 \times -4) = 3(16) = \textcircled{48}$

(c) $y^2 - 5x = (3)^2 - 5(2) = 9 - 10 = \textcircled{-1}$

(d) $3a^2 - 2ab + b^2 = 3(2)^2 - 2(2)(-1) + (-1)^2 = 12 + 4 + 1 = \textcircled{17}$

(e) $3m^2n - 5n^2 = 3(2)^2(-3) - 5(-3)^2 = 3(2 \times 2)(-3) - 5(-3 \times -3) = -36 - 45 = \textcircled{-81}$

Exercise #10, page 31

1. (a) $3, \ r^2s$ (b) $-1, \ b^2$ (c) $\frac{1}{3}, \ \frac{mn}{s}$

(d) $-6, \ \frac{1}{y}$ (e) $1, \ \frac{ab}{c}$ (f) $\frac{2}{3}, \ \frac{y}{x}$

$$\left.\begin{array}{ll}
2.\ (a) & \frac{ab^2}{3c^2} = \frac{1}{3} \times a \times b \times b \times \frac{1}{c} \times \frac{1}{c} \\[1em]
(b) & -6x^2y^2 = -6 \times x \times x \times y \times y \\[1em]
(c) & \frac{mn^3}{5} = \frac{1}{5} \times m \times n \times n \times n \\[1em]
(d) & \frac{3a^2bc}{7} = \frac{3}{7} \times a \times a \times b \times c \\[1em]
(e) & -y^4z = \underset{\nwarrow \text{NUMERICAL COEFFICIENT IS } -1.}{-1} \times y \times y \times y \times y \times z
\end{array}\right\} \text{FACTORS MAY BE LISTED IN ANY ORDER.}$$

Exercise #10, continued

3. (a) $3mn + 2mn = \underline{5mn}$

(b) $6a^2 - 5a^2 = \underline{a^2}$

(c) $3b - 7b = \underline{-4b}$

(d) $-2xy^2 - 4xy^2 = \underline{-6xy^2}$

(e) $4abc - 4abc = \underline{0}$

(f) $-6m + 3m = \underline{-3m}$

(g) $\sqrt{2}y + 5\sqrt{2}y = \underline{6\sqrt{2}y}$

(h) $\sqrt{3}y - 7\sqrt{3}y = \underline{-6\sqrt{3}y}$

(i) $3a + 2b - b + 4a$
$= (3a + 4a) + (2b - b)$
$= \underline{7a + b}$

(j) $x^2 - xy + y^2 - 3y^2 - xy + 4x^2$
$= (x^2 + 4x^2) + (-xy - xy) + (y^2 - 3y^2)$
$= \underline{5x^2 - 2xy - 2y^2}$

(k) $3m + 4n - 5m - n + 2n^2$
$= (3m - 5m) + (4n - n) + 2n^2$
$= \underline{-2m + 3n + 2n^2}$

(l) $xy - y^2 + y - 3xy + 3y^2 - 4y$
$= (xy - 3xy) + (-y^2 + 3y^2) + (y - 4y)$
$= \underline{-2xy + 2y^2 - 3y}$

(m) $5x^2 - 6x + 8 - 8x^2 + 7x + 11 + 2x^2 - 3x - 4$
$= (5x^2 - 8x^2 + 2x^2) + (-6x + 7x - 3x) + (8 + 11 - 4)$
$= \underline{-x^2 - 2x + 15}$

Exercise #11, page 34

1. (a) $x^5 \times x^2 = x^{5+2} = \underline{x^7}$

(b) $m^{-4} \times m^2 = m^{-4+2} = \underline{m^{-2}}$

(c) $2y^3 \times -5y = -10y^{3+1} = \underline{-10y^4}$

(d) $(-ab)(3a^2) = -3a^{1+2}b = \underline{-3a^3b}$

(e) $(-7x^2y)(-2xy^2) = 14x^{2+1}y^{1+2} = \underline{14x^3y^3}$

(f) $(6y^2)(-4y^2) = -24y^{2+2} = \underline{-24y^4}$

(g) $(\frac{1}{2}xy)(-2xy^3) = -1x^{1+1}y^{1+3} = \underline{-x^2y^4}$

(h) $(6cd)(-5c^2)(3d^2) = -90c^{1+2}d^{1+2} = \underline{-90c^3d^3}$

(i) $(-4m^2n^3)(-mn^4)(-2m^3n) = -8m^{2+1+3}n^{3+4+1} = \underline{-8m^6n^8}$

Exercise #12, page 36

1. (a) $\frac{y^4}{y^2} = y^{4-2} = y^2$

(b) $\frac{x}{x^3} = \frac{1}{x^{3-1}} = \frac{1}{x^2}$

(c) $\frac{m^9}{m} = m^{9-1} = m^8$

(d) $\frac{r^2}{r^5} = \frac{1}{r^{5-2}} = \frac{1}{r^3}$

(e) $\frac{24y^4}{-3y^3} = -8y^{4-3} = -8y$

(f) $\frac{-6b}{12b^2} = \frac{-1}{2b^{2-1}} = \frac{-1}{2b}$

Exercise #12, continued

(g) $\frac{-15a^3b^2c^2}{12ab} = \frac{-15}{12} \times \frac{a^3}{a} \times \frac{b^2}{b} \times c^2 = \frac{-5}{4} \times a^{3-1} \times b^{2-1} \times c^2 = \frac{-5a^2bc^2}{4}$

(h) $\frac{4ab^2c^2}{2b^4c} = \frac{4}{2} \times a \times \frac{b^2}{b^4} \times \frac{c^2}{c} = 2 \times a \times \frac{1}{b^{4-2}} \times c^{2-1} = \frac{2ac}{b^2}$

(i) $\frac{6m^5n^2}{-2m^3n^3} = \frac{6}{-2} \times \frac{m^5}{m^3} \times \frac{n^2}{n^3} = -3 \times m^{5-3} \times \frac{1}{n^{3-2}} = \frac{-3m^2}{n}$

Exercise #13, page 43

1. (a) $(-7ab)^2 = (-7)^2a^2b^2 = 49a^2b^2$

(b) $(3y)^3 = 3^3y^3 = 27y^3$

(c) $\left(\frac{-6}{x}\right)^2 = \frac{(-6)^2}{x^2} = \frac{36}{x^2}$

THE EXPONENT 4 APPLIES TO EVERY FACTOR IN THE NUMERATOR AND DENOMINATOR.

(d) $\left(\frac{2ab}{3c}\right)^4 = \frac{(2ab)^4}{(3c)^4} = \frac{2^4a^4b^4}{3^4c^4} = \frac{16a^4b^4}{81c^4}$

(e) $\left(\frac{-2m^3}{5n^2}\right)^2 = \frac{(-2m^3)^2}{(5n^2)^2} = \frac{(-2)^2(m^3)^2}{5^2(n^2)^2} = \frac{4m^6}{25n^4}$

Exercise #14, page 45

1. (a) $3y^{-5} = 3 \times \frac{1}{y^5} = \frac{3}{y^5}$

(b) $b^{-2} = \frac{1}{b^2}$

(c) $\frac{1}{x^{-3}} = \frac{1}{\frac{1}{x^3}} = x^3$

(d) $\frac{1}{4y^{-7}} = \frac{1}{4 \times \frac{1}{y^7}} = \frac{1}{\frac{4}{y^7}} = \frac{y^7}{4}$

(e) $\frac{3ab^{-1}}{c^{-1}} = \frac{3a \times \frac{1}{b}}{\frac{1}{c}} = \frac{3a}{b} \div \frac{1}{c} = \frac{3a}{b} \times \frac{c}{1} = \frac{3ac}{b}$

Exercise #14, continued

2. (a) $(2y)^{-3} = \frac{1}{(2y)^3} = \frac{1}{2^3y^3} = \frac{1}{8y^3}$

(or)

$(2y)^{-3} = 2^{-3} \times y^{-3} = \frac{1}{2^3} \times \frac{1}{y^3} = \frac{1}{8y^3}$

(b) $\frac{x^{-2} \times x^5}{x^2} = \frac{x^{-2+5}}{x^2} = \frac{x^3}{x^2} = x^{3-2} = x$

(c) $\frac{(ab)^2}{b^{-3} \times a^2} = \frac{a^2b^2}{b^{-3} \times a^2} = a^{2-2}\,b^{2-(-3)} = a^0\,b^5 = b^5$

(d) $\frac{(a^2)^{-4} \times (a^5)^3}{(a^{-2})^{-2}} = \frac{a^{-8} \times a^{15}}{a^4} = a^{-8+15-4} = a^3$

(e) $\left(\frac{3a^2}{4b}\right)^{-3} = \frac{1}{\left(\frac{3a^2}{4b}\right)^3} = \left(\frac{4b}{3a^2}\right)^3 = \frac{4^3\,b^3}{3^3(a^2)^3} = \frac{64b^3}{27a^6}$

(or)

$\left(\frac{3a^2}{4b}\right)^{-3} = \frac{3^{-3} \times (a^2)^{-3}}{4^{-3} \times b^{-3}} = \frac{\frac{1}{3^3} \times \frac{1}{a^6}}{\frac{1}{4^3} \times \frac{1}{b^3}} = \frac{4^3b^3}{3^3a^6} = \frac{64b^3}{27a^6}$

End of Lesson C

Lesson D

POLYNOMIALS

Topic One: Defining and Classifying Polynomials

A. Definition of a Polynomial

A POLYNOMIAL is an expression which consists of a single term or a sum of terms. Each term is a real number, a positive integral power of a variable, or a product each of whose factors is a real number or a positive integral power of a variable.

Although every polynomial is considered to be a sum, the signs of addition are not always indicated. For example, the expression $(-2x) + (+3y) + (-5z)$ is a polynomial with three terms. These three terms are $-2x$, $+3y$, and $-5z$. Because it is cumbersome to write the polynomial in the manner indicated, the signs of addition are usually omitted, and the polynomial is written in the form $-2x + 3y - 5z$. Although the addition signs are not written, they are understood, and the terms of the polynomial are still considered to be $-2x$, $+3y$, and $-5z$.

Below are some examples of polynomials. Fill in any blanks that appear in the discussion of these examples.

EXAMPLE 1: $\frac{3}{2}x + 5y^2 + 4xy^3 + 7$

Discussion

This polynomial consists of four terms. The first term, $\frac{3}{2}x$, is the product of the real number ______ and the variable ______ raised to the first power. The second term, $5y^2$, is the product of the real number ______ and the variable ______ raised to the ______ power. The third term, $4xy^3$, is the product of the real number ______, the variable x raised to the ______ power, and the variable y raised to the ________ power. The fourth term, ______, is a real number.

EXAMPLE 2: $\sqrt{3}t^4 - 7t^2$

Discussion

If this polynomial were written in the form $\sqrt{3}t^4 + (-7t^2)$, it could be regarded as a sum. This polynomial consists of ______ terms. The first term, $\sqrt{3}t^4$, is the product of the real number ______ and the variable ______ raised to the ______ power. The second term, $-7t^2$, is the product of the real number ______ and the variable ______ raised to the ______ power.

Our definition of a polynomial excludes certain kinds of variable expressions from being classified as polynomials. The following expressions are not polynomials.

1. $5x + \sqrt{-2}$

Expressions which contain numbers that do not belong to set R are not polynomials. This expression is not a polynomial because it contains the term $\sqrt{-2}$ which is not a real number.

2. $3\sqrt{x} - 5x + 7$

Expressions which contain roots of variables are not polynomials. This expression is not a polynomial because the first term contains the factor $\sqrt{x}$ which is a root of a variable. (Remember that $\sqrt{x}$ only represents a real number for non-negative replacements for x.)

3. $-3x^3 + 5x^{-2} + y^{\frac{1}{4}}$

Expressions which contain negative or fractional powers of a variable are not polynomials. This expression is not a polynomial since the second term contains the factor x^{-2} which is a negative power of the variable x. Also, the third term is a fractional power of the variable y.

4. $\frac{5}{x} - 3$

Expressions which contain a variable in the denominator are not polynomials. This expression is not a polynomial since the variable x appears in the denominator of the first term.

Self-correcting Exercise #1

Answers may be found on page 55 of this lesson.

1. Decide whether or not each of the following expressions is a polynomial. Write "yes" in the blank if it is and "no" if it isn't.

(a) $w + \sqrt{6}$ yes

(b) $\frac{6}{x^4} - 2x^3$ ______

(c) 7 ______

(d) $\frac{1}{2}x^3 + \frac{2}{3}x - \frac{4}{5}$ ______

(e) $x^2 - 3x - \sqrt{x} + 1$ ______

(f) $-3x + 2x^{-1}$ ______

(g) $5y^3$ ______

(h) $5mn - 3m^2 + \sqrt{-1}$ ______

(i) $\frac{x^2 + 5}{x}$ ______

(j) $4x^{\frac{2}{3}}$ ______

(k) $x - 3$ ______

(l) $\frac{-2}{x^2} + 5$ ______

2. List the terms in each of the following polynomials.

(a) $-5x^2 + 3x - 9$

$-5x^2, 3x, -9$

(b) $\frac{2}{3}xy^2z$

______

(c) $y - 7$

______

(d) $5 - a^2b - b^3$

______

(e) $\sqrt{2}x + \sqrt{3} - 7x^2 + 4x^3$

______

(f) $abc - 2b^2 + c$

______

B. Classifying Polynomials

For easy reference, polynomials are often classified according to the number of different terms they contain. Like terms should always be combined before the terms in a polynomial are counted.

EXAMPLES:

1. $2xy + z$ is a two-term polynomial.

2. $3a - 5b + 2c - 8d$ is a four-term polynomial.

3. $x^2 + 2x - 3x^2 + 5x + 7$ contains some like terms. When like terms are combined, this polynomial can be written as $-2x^2 + 7x + 7$ and can be classified as a three-term polynomial.

Polynomials with one, two, or three terms are given special names.

A MONOMIAL is a polynomial with one term. The following expressions are all monomials.

$$8, \quad -x, \quad 3y^2, \quad \frac{-4xy}{3}, \quad 9x^2y^3z$$

A BINOMIAL is a polynomial with two terms. The following expressions are all binomials.

$$y + 3, \quad 2x^2 + x, \quad 5a^2b - 9ab^2, \quad x^5 + \sqrt{2}$$

A TRINOMIAL is a polynomial with three terms. The following expressions are all trinomials.

$$3a + 5b - 2c, \quad x^2 - 5x + 9, \quad x^5 + 7x^3 - 4x$$

It is customary to refer to a polynomial of four or more terms simply as a "polynomial" and state the number of terms it contains.

Classify each of the following polynomials according to its number of terms. (Combine any like terms where possible before classifying.)

1. $-3x^2 + y$ binomial
2. $\frac{-xyz}{5}$ __________
3. $3m + 3n - 4$ __________
4. $3x + 5y + y^2 - 2x + 7 = x + 5y + y^2 + 7$ polynomial (4 terms)
   LIKE TERMS
5. $a - b$ __________
6. $a^2 + b^2 + c^2 - 3b^2 - 5$ __________
7. $3a^2b - 7a^2b$ __________
8. $3a - 5a + b + 6b$ __________
9. $6mn^2$ __________
10. $5x^2 - 7x + 4x - 9$ __________

A second method of classifying polynomials involves stating the number of variables the polynomial contains. For example, each of the polynomials below can be classified as a POLYNOMIAL IN ONE VARIABLE.

$$x, \quad 3t + 5, \quad 2m^2 + m - 3, \quad y^4 - 3y^2 + 5y + 1$$

Which polynomial listed above is a trinomial? ____________

Which is a monomial? ____________

Which is a binomial? ____________

Give an example of a binomial in the one variable z. (Make up your own example.) ____________

The following polynomials all involve more than one variable.

$$x^2 + y, \quad xy + z, \quad 3tuv + w^2 + v^2$$

The first expression is a polynomial in the two variables x and y. The second expression is a polynomial in the three variables x, y, and z.

How many different variables are involved in the third polynomial? ____________

Which expression is a trinomial? ____________

Give an example of a monomial in the three variables a, b, and c. ____________

Classify each of the following polynomials with respect to the number of variables involved.

1. $3xy^3 + x^2y + 5$ polynomial in two variables
2. $3a + 2$ ____________
3. $4yz + 6z^2 + 7xyz$ ____________
4. $6ab + b^2 + a^2b - 2a^2$ ____________
5. $mn^2 + 3xy + 7y^2$ ____________
6. $3a^2 + 2b^2 - 5ac^2$ ____________
7. $7x^3 + 3x^2 - 5x + 2$ ____________

Polynomials can also be classified according to their degree. The degree of a polynomial depends upon the degree of its individual terms. The degree of each term can be determined by applying the following rules.

1. A term which is a non-zero real number has degree zero. For example, the term -5 has degree zero.

2. A term which contains only one variable factor has a degree which is equal to the exponent of the variable. For example, the term $7y^3$ has degree 3 since the variable y has an exponent of 3.

3. A term which contains more than one variable factor has a degree which is equal to the sum of the exponents of the variables. For example, the term $2ab^2c^3$ has degree 6 since the sum of the exponents of the variables is $1 + 2 + 3 = 6$.

The term with the highest degree in a polynomial determines the degree of that polynomial.

EXAMPLE: What is the degree of the polynomial $15x^4 - 7x^3y^2 + 2xy$?

Solution

In the polynomial $15x^4 - 7x^3y^2 + 2xy$, the first term has degree 4, the second term has degree 5, and the third term has degree 2. The second term has the highest degree. Since the second term has degree 5, the polynomial is of degree 5.

State the degree of each of the following polynomials.

1. $2x^2 + 5xy + y^2$ degree 2
2. $-5abc^4$ ________
3. $6m^2n + 3mn + 5n^2 + 3n$ ________
4. $4x - 3$ ________
5. $2x^2 + 3x^2y^2 + 9y^2$ ________
6. $3x^2 + 4x - 9$ ________
7. $5x^3y^3z$ ________
8. $4ab + 5b$ ________

Polynomials with degree zero, one, two, three, and four are given special names.

A polynomial with degree zero is called a CONSTANT POLYNOMIAL. The following expressions are all constant polynomials:

-12, 4, $\frac{3}{4}$, $\sqrt{2}$, $2x^0$

(REMEMBER THAT $X^0 = 1$.)

Could each of these constant polynomials be classified as a monomial? ____________.

A polynomial with degree one is called a LINEAR POLYNOMIAL. The following expressions are all linear polynomials:

$-3y$, x, $a + 3$, $\frac{2}{3}a - \frac{1}{2}b + c$, $x - 5y$

Which of these linear polynomials are monomials? $-3y, x$

Which is a trinomial? ____________

Which is a polynomial in two variables? ____________

Which is a linear binomial in one variable? ____________

A polynomial with degree two is called a QUADRATIC POLYNOMIAL. The following expressions are all quadratic polynomials:

$\frac{1}{2}x^2$, $4xy$, $7y^2 - 3y$, $a^2 + 3ab + b^2$,

$2yz - y$, $12x^2 + 17x + 15$, $3a + 5bc + 6c$, $2y^2 - 3$

Which quadratic polynomials above are binomials?

____

Which is a quadratic trinomial in one variable? ____________

Which is a quadratic trinomial in two variables? ____________

Which is a quadratic trinomial in three variables? ____________

Which is a quadratic monomial in two variables? ____________

A polynomial with degree three is called a CUBIC POLYNOMIAL. The following expressions are all cubic polynomials:

$7y^3$, $2xy^2$, $4x^3 - 3xy^2$, $5xyz + yz + z$,

$2a^3 - 5a^2 + 3$, $2a^2b + b$, $5x^3 - 3x^2 + 7x - 9$

Which expressions above are cubic binomials in two variables?

____

Which expression above is a cubic trinomial in one variable?

____

A polynomial with degree four is called a QUARTIC POLYNOMIAL. The following expressions are all quartic polynomials:

$\sqrt{2}x^4$, $9x^3y$, $5a^2bc - 3a^2 + 2b^2c$, $2x^2y^2 + 8$

$5x^4 - 3x^2 + 7$, $7xy^2z + 2x - 3z^2 + y^3$, $x^4 + x^3 + x^2 + x + 1$

In the quartic polynomial $5a^2bc - 3a^2 + 2b^2c$, which term is of degree four? __________

Which expression above is a quartic binomial in two variables?

____

A fourth method of classifying polynomials is concerned with the numerical coefficients of the terms of the polynomial. If the numerical coefficients belong to a given set of numbers, the expression is said to be a polynomial over that set of numbers.

EXAMPLES:

1. $2x^3 + 5x - 20$ is a POLYNOMIAL OVER THE INTEGERS since all its coefficients belong to set I. (Note that this expression is also a polynomial over the rationals and the reals since $I \subseteq Q \subseteq R$, but we usually use the most descriptive classification.) "IS A SUBSET OF"

2. $\frac{2}{3}y^2 - 2y + \frac{1}{4}$ is a POLYNOMIAL OVER THE RATIONALS since all its coefficients belong to set Q. (Since $Q \subseteq R$, it is also a polynomial over the reals.)

3. $\pi x^2 + 5xy - \sqrt{3}y^2$ is a POLYNOMIAL OVER THE REALS since all its coefficients belong to set R.

Is $x + \frac{y}{2}$ a polynomial over the integers? __________

Why? ____

Is $x^2 - 3x + 7y + 2z^2$ a polynomial over the rationals? __________

Why? ____

Self-correcting Exercise #2

Answers may be found on page 55 of this lesson.

1. For each polynomial, state the degree of each term and then state the degree of the polynomial.

   (a) $3xy + 2y^3 - xy^4$

   (i) The first term is of degree _____, the second term is of degree _____, and the third term is of degree _____.

   (ii) The polynomial is of degree _____.

   (b) $a^2 - abc + b^2d + 8c - 5$

   (i) The first term is of degree _____, the second term is of degree _____, the third term is of degree _____, the fourth term is of degree _____, and the fifth term is of degree _____.

   (ii) The polynomial is of degree _____.

2. For each polynomial, state the number of terms it contains, the number of variables involved, and its degree.

	No. of Terms	No. of Variables	Degree
(a) $xy + 2x^2y^2 + 3y + x^3$			
(b) $-7x^3y^3z$			
(c) $a^4b^4 - 3a^3b^4 + 7b^5$			
(d) $ab^2c + 4a^2bc^2 - b^2c^2 + 8$			
(e) $3xy^5 - x^2y^3$			

3. Classify each of the following polynomials according to its coefficients. (Give the most descriptive classification.)

(a) $x^2 + \frac{1}{2}x + 4$ polynomial over the rationals

(b) $y^2 - x$ ________________

(c) $2\pi + 3x^4$ ________________

(d) $t^2 + 3t - 2$ ________________

(e) $\frac{x^2}{5} - 3x + 7$ ________________

4. In part (i), classify each polynomial as being a monomial, binomial, or trinomial. In part (ii), classify each polynomial as being a constant, linear, quadratic, cubic, or quartic polynomial.

(a) $8x - 5$

(i) ________________

(ii) ________________

(b) $-3x^3 + 4x - 5$

(i) ________________

(ii) ________________

(c) $-5m^0$

(i) ________________

(ii) ________________

(d) $7m^3n - mn^2$

(i) ________________

(ii) ________________

(e) $x^2 + 2xy - 3y^2$

(i) ________________

(ii) ________________

(f) $\sqrt{3}y$

(i) ________________

(ii) ________________

(g) $4a^2 + 3a$

(i) ________________

(ii) ________________

(h) $2abc + 3b^2$

(i) ________________

(ii) ________________

C. Polynomials in One Variable

A POLYNOMIAL IN ONE VARIABLE consists of terms which are real numbers or the product of real numbers and positive integral powers of one particular variable. For example, since the polynomial $5x^3 - 7x + 2$ contains only the one variable x it is a polynomial in one variable. On the other hand, the polynomial $5x^2 + 2xy + 3y^2$ contains both the variable x and the variable y. Therefore, it is not a polynomial in one variable.

A polynomial in one variable is written in STANDARD FORM when it satisifies the following conditions:
1. All like terms have been combined.
2. The terms are written in such a manner that the exponents of the variable decrease from left to right.

For example, the expression $3x^2 - 5x + 9x^3 + 15x + 12 - 6x^2$ is a polynomial in one variable, but it is not written in standard form.

When like terms are combined and the exponents written in decreasing order from left to right, the standard form of the polynomial is $9x^3 - 3x^2 + 10x + 12$.

A LINEAR POLYNOMIAL IN ONE VARIABLE can always be written in the standard form $ax + b$, where x is a variable, a and b are real numbers, and $a \neq 0$.

The stipulation that $a \neq 0$ must be made above to ensure that the first degree term in the polynomial does not disappear. Note that if $a = 0$, the term "ax" equals zero, and the polynomial would be constant rather than linear.

Once a linear polynomial in one variable is written in the standard form $ax + b$, the values of "a" and "b" can be readily stated.

"a" is the numerical coefficient of the first-degree term.
"b" is the constant term.

The following expressions are linear polynomials in one variable (written in standard form).

1. $4x - 7$

   In this case, the variable is x. $a = 4$ since the first-degree term is $4x$ and its numerical coefficient is 4. $b = -7$ since the constant term is -7.

2. $\frac{-3}{4}y$

   In this case, the variable is y. $a = \frac{-3}{4}$ since the first-degree term is $\frac{-3}{4}y$ and its numerical coefficient is $\frac{-3}{4}$. $b = 0$ since there is no constant term.

Write the linear polynomial $3x - 5x + 7 - 12x + 8x - 4 + 3$ in standard form. ____________________ In this case, a = _____ and b = _____.

Write the linear polynomial $7 + 6y - 4 - 7y - 3$ in standard form. ____________________ In this case, a = _____ and b = _____.

A QUADRATIC POLYNOMIAL IN ONE VARIABLE can always be written in the standard form $ax^2 + bx + c$, where x is a variable; a, b, and c are real numbers; and $a \neq 0$.

The stipulation that $a \neq 0$ must always be made to ensure that the second degree term in the polynomial does not disappear. Note that if $a = 0$, the term "ax^2" equals zero, and the polynomial would linear rather than quadratic.

Once a quadratic polynomial in one variable is written in the standard form $ax^2 + bx + c$, the values of a, b, and c can be readily stated.

"a" is the numerical coefficient of the second-degree term.
"b" is the numerical coefficient of the first-degree term.
"c" is the constant term.

The following expressions are quadratic polynomials in one variable (written in standard form).

1. $-6x^2 + 3x - 7$
   In this case, the variable is x.
   $a = -6$ since the second-degree term is $-6x^2$ and its numerical coefficient is -6.
   $b = 3$ since the first-degree term is $3x$ and its numerical coefficient is 3.
   $c = -7$ since the constant term is -7.

2. $\frac{1}{2}r^2 + \frac{3}{2}$
   In this case, the variable is r.
   $a = \frac{1}{2}$ since the second-degree term is $\frac{1}{2}r^2$ and its numerical coefficient is $\frac{1}{2}$.
   $b = 0$ since there is no first-degree term.
   $c = \frac{3}{2}$ since the constant term is $\frac{3}{2}$.

Write the quadratic polynomial $5x + 3x^2 - 5x^2 + 7x - 4x + 3 - 2x^2 + 3x^2 - 9$ in standard form. ____________________ In this case, a = _______, b = _______, c = _______.

Write the quadratic polynomial $8 - 2y + 3y^2 - 8 - y$ in standard form ____________________ In this case, a = _____, b = _____, and c = _____.

A CUBIC POLYNOMIAL IN ONE VARIABLE can always be written in the standard form $ax^3 + bx^2 + cx + d$, where x is a variable; a, b, c, and d are real numbers; and $a \neq 0$.

The following expressions are cubic polynomials in one variable (written in standard form).

1. $5m^3 - 10m^2 + 3m - 2$
   In this case, the variable is m.
   $a = 5$, $b = -10$, $c = 3$, and $d = -2$.

2. $-2y^3 + 3y - 5$
   In this case, the variable is y.
   $a = -2$, $b = 0$, $c = 3$, and $d = -5$.
   (There is no second-degree term.)

Write the cubic polynomial $4y^2 - 3y - 4y^3 + 2y^2 + 7y - 9 + 5y^3 + 8 - 4y$

in standard form. ______________________ In this case, a = ______,

b = ______, c = ______, and d = ______.

A QUARTIC POLYNOMIAL IN ONE VARIABLE can always be written in the standard form $ax^4 + bx^3 + cx^2 + dx + e$ where x is a variable; a, b, c, d, and e are real numbers; and $a \neq 0$.

The following expressions are quartic polynomials in one variable (written in standard form).

1. $x^4 - 3x^3 - 7x^2 - x - 9$
   In this case, the variable is x.
   $a = 1$, $b = -3$, $c = -7$, $d = -1$, $e = -9$

2. $5t^4 - 7t^2 + 3t$
   In this case, the variable is t.
   $a = 5$, $b = 0$, $c = -7$, $d = 3$, $e = 0$
   (There is no third-degree term.) (There is no constant term.)

Write the quartic polynomial $4m^3 + 7m^2 - 5m^4 + 6 - 3m^2 - 8m^3$ in

standard form. ______________________ In this case, a = ______,

b = ______, c = ______, d = ______, and e = ______.

Self-correcting Exercise #3

Answers may be found on page 56 of this lesson.

Write each of the following polynomials in one variable in standard form. Then, classify the polynomial as being linear, quadratic, cubic, or quartic and state the values of the numerical coefficients.

EXAMPLE: $-4x^2 + 3x + 5x^3 + 9x - 10$

Standard form $5x^3 - 4x^2 + 12x - 10$

Classification cubic $(ax^3 + bx^2 + cx + d)$

Numerical coefficients $a = 5, \ b = -4, \ c = 12, \ d = -10$

1. $7 + 3x - 9 - 7x$

Standard form ____________________

Classification ____________________

Numerical coefficients ____________________

2. $-12 + 2x^4 + 13 - 7x^3$

Standard form ____________________

Classification ____________________

Numerical coefficients ____________________

3. $6x - 7x + x^2$

Standard form ____________________

Classification ____________________

Numerical coefficients ____________________

4. $\frac{1}{3}x - \frac{3}{5}x^3 + \frac{2}{5}$

Standard form ____________________

Classification ____________________

Numerical coefficients ____________________

5. $7 - 8x^2$

Standard form ______________________________

Classification ______________________________

Numerical coefficients ______________________________

EXERCISE - Polynomials

1. Fill in the blanks.

(a) Since the polynomial $-3xy + y^2 + 2x$ has ________________ different terms, it is called a ________________.

(b) The algebraic expression $\frac{5}{y} + 7$ is not a polynomial because a ________________ appears in the denominator.

(c) The terms of the polynomial $-ax^3 + 3bx^2 - 4x - 1$ are ______, ______, ______, and ______.

(d) The degree of the monomial $-3a^3b^3c$ is ____________.

(e) A ____________________ is a polynomial with two terms.

(f) The polynomial $xy^2 + x^3y^3 + 2y^5$ is of degree ______________.

(g) A polynomial of degree one is called a __________________ polynomial.

(h) The algebraic expression $2x^{-3} + \sqrt{-5}$ is not a polynomial because the first term contains a ________________ exponent and the second term is not a ______________________ number.

(i) The degree of a polynomial depends upon its term with the ______________________ degree.

(j) A polynomial in one variable is written in standard form when all ______________________ terms have been combined and the exponents of the variable ______________________ from left to right.

(k) Any cubic polynomial in one variable can be written in the standard form ____________________.

(l) The polynomial $3m^2n^2 + 5mn + 2n$ is a ________________ polynomial because it has degree ________________.

(m) In the polynomial $-x^3 - 8$, the numerical coefficient of the third degree term is ________ and the constant term is __________.

(n) $ax^2 + bx + c$ is the standard form of a ____________ polynomial in ________________ variable.

(o) The polynomial $-\sqrt{7}$ is a ____________________ polynomial because it has degree ________________.

2. Fill in the blanks in the following chart. For each polynomial, you must give the number of terms involved, the number of variables involved, and the degree of the polynomial.

Polynomial	Number of Terms	Number of Variables	Degree
(a) $x^3 - 3xy + 5y^2$	3(trinomial)	2	3(cubic)
(b) $3x^2yz$			
(c) $5 + 2ab$			
(d) -7			
(e) $-2y + 3$			
(f) $x^2 - 4xy + 2y^2$			
(g) $x^4 - x^2$			
(h) $-5a^3 + 2ab + 3b^2 - 7$			
(i) $5y^2 + 7$			
(j) $x^4 - 5x^2 + 4$			
(k) $m^2n^2 - 1$			

3. From the list of algebraic expressions given below, pick out those expressions which are not polynomials and explain why they are not.

$-6x^2 + \sqrt{3}$

$4x^{-2}$

$3m^2 - 3mn$

$\frac{1}{y} + 7$

$5\sqrt{ab} + b - 7$

$x - 9$

$2xy + y + x^{\frac{1}{2}}$

$\frac{4x^3 + 2y^2}{z}$

4. Give an example of each of the following kinds of polynomials.

(a) binomial ________________

(b) four-term polynomial ________________

(c) monomial of degree five ________________

(d) cubic trinomial in two variables ________________

(e) constant polynomial ________________

(f) linear binomial in one variable ________________

(g) quadratic trinomial in one variable ________________

(h) quartic monomial in one variable ________________

5. Classify each of the following polynomials according to its numerical coefficients. (Give the most descriptive classification.) Review #3 on page 10.

(a) $x + 2y^2 - 3xy$ ____________________

(b) $1.5x^2 - 0.3y$ ____________________

(c) $\pi x^2 + y - 2y^2$ ____________________

(d) $-mn$ ____________________

6. Fill in the blanks in the following chart. Put each polynomial in standard form, classify it in terms of degree, and then state the values of the numerical coefficients.

Polynomial in One Variable	Standard Form	Classification (Degree)	Numerical Coefficients				
			a	b	c	d	e
$x^2 - 3x^3 + 5$	$-3x^3 + x^2 + 5$	cubic $(ax^3 + bx^2 + cx + d)$	-3	1	0	5	—
$7 - 3x - 2x$							
$2y^3 - 3y^4 + y^2 - 1$							
$8 - 3x^2 + 9$							
$2x^2 - 5x^3 + 4$							
$3 + 2y + 6y^2$							
$8y^3 + 3y - 5 + 2y^2$							
$y^2 - y^3 + y^4 - 1$							
$-2x^2$							
$3 - x + 1$							
$4x^2 - 5x - 5x^2$							
$\frac{2}{3}x$							

7. Decide whether each of the following statements is true or false.

(a) Every polynomial is either a monomial, binomial, or trinomial. *false*

(b) The degree of the polynomial $\sqrt{5}$ is 0. ________

(c) Every monomial is a polynomial. ________

(d) A binomial is a polynomial of degree two. ________

(e) The numerical coefficient of $-t$ is 1. ________

(f) The degree of $5x + yz$ is 2. ________

(g) Every monomial is a polynomial in one variable whose degree is one. ________

(h) $2x + 3$ is a polynomial over the integers. ________

(i) 9 is a polynomial. ________

(j) $3y^3 + 7y - 5$ is a cubic trinomial. ________

(k) A polynomial may have 500 terms. ________

(l) $\frac{1}{x^2}$ is a monomial. ________

(m) $2y + 3$ is a linear binomial in one variable. ________

(n) $5 + 2t^2 - 7t$ is a quadratic polynomial in one variable written in standard form. ________

(o) $\frac{x + y}{a + b}$ is a linear binomial in four variables. ________

(p) $3x^2 - \sqrt{5}xy + 2y^2$ is a polynomial over the rationals. ________

(q) The second term of the polynomial $x^3 - 3y^2 + 5y + 6x$ is $3y^2$. ________

Topic Two: Operating With Polynomials

A. Addition of Polynomials.

Two or more polynomials can be combined by addition by first using the commutative and associative properties of addition to group like terms together, and then using the distributive property to add numerical coefficients of like terms. The sum of polynomials is always a polynomial.

EXAMPLE: Combine the polynomials $3x^2y + 4x - 7$, $-7x^2y - 3y$, and $5x - 8y + 10$ by addition.

Solution

$$(3x^2y + 4x - 7) + (-7x^2y - 3y) + (5x - 8y + 10)$$

Remove brackets, and write as a single polynomial.

$$= 3x^2y + 4x - 7 - 7x^2y - 3y + 5x - 8y + 10$$

Group like terms together.

$$= (3x^2y - 7x^2y) + (4x + 5x) + (-3y - 8y) + (-7 + 10)$$

AT THIS POINT COUNT THE TERMS TO MAKE SURE YOU HAVEN'T MISSED ANY.

Add numerical coefficients of like terms.

$$= (3 - 7)x^2y + (4 + 5)x + (-3 - 8)y + (-7 + 10)$$

$$= \underline{\underline{-4x^2y + 9x - 11y + 3}}$$

(Note that the sum is also a polynomial.)

Add the following polynomials by combining like terms.

1. $(7x^2 - x + 2) + (3x^2 - 10x) + (x - 4) + (-5x^2 + 8)$

$= 7x^2 - x + 2 + 3x^2 - 10x + x - 4 - 5x^2 + 8$

$= (7x^2 + 3x^2 - \underline{\qquad}) + (-x - \underline{\qquad} + \underline{\qquad}) + (2 - \underline{\qquad} + \underline{\qquad})$

$= \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$

2. $(x^2 + 3xy^2 + 5xy - 2y^2) + (-7xy + 4x^2y) + (3x^2 - 5xy^2 - 2y^2)$

$=$

For convenience in adding polynomials, we may use a vertical form. In this method, we arrange terms in a vertical fashion in such a way that like terms lie below each other. For each set of like terms we add the numerical coefficients and attach the common variable factor to this sum. (The distributive property allows us to do this.)

EXAMPLE: Add the polynomials $(3x^3 - 4x^2 + 7x + 5)$, $(6x^3 + 2x^2 - 8)$, and $(9x^2 + 2x + 1)$, using the vertical form.

Solution

$$\begin{array}{rrrr} 3x^3 & -\ 4x^2 & +\ 7x & +\ 5 \\ 6x^3 & +\ 2x^2 & & -\ 8 \\ & 9x^2 & +\ 2x & +\ 1 \\ \hline 9x^3 & +\ 7x^2 & +\ 9x & -\ 2 \end{array} \quad \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{Like terms have been placed under each other.}$$

Find the following sums.

1. $$\begin{array}{rrr} x & -\ y & +\ z \\ 2x & -\ 3y & -\ 2z \\ \hline \end{array}$$

2. $$\begin{array}{rrr} 2n & -\ 5k & +\ 6 \\ n & -\ k & -\ 8 \\ \hline \end{array}$$

3. $$\begin{array}{rrr} a^2 & +\ 5a & -\ 9 \\ -3a^2 & -\ 10a & +\ 7 \\ & 4a & -\ 3 \\ \hline \end{array}$$

Self-correcting Exercise #4

Answers may be found on page 56 of this lesson.

1. Add the following polynomials by removing parentheses and combining like terms.

   (a) $(-2a^2 + 3a + 4) + (5a^2 - 9) + (-7a + 12)$

   $=$

   (b) $(5y^6 - 2y^2 + y) + (-2y^2 - 3y^5 + y) + (y^5 - y^4)$

   $=$

(c) $(a + b + c) + (2a - 3b) + (-6a - 2c) + (4b + 3c) + (-3a - b + 2c)$

=

2. Add the given polynomials using the vertical form. (Remember to place like terms under each other.)

(a) $(2ay - 3y + 2)$, $(4ay - 7)$, $(-2ay - 6y + 3)$

(b) $(4x^5 + 2x^3 + x^2 - 1)$, $(-8x^2 + 3)$, $(-9x^5 + 2x^3 + 6)$

B. The Additive Inverse of a Polynomial

In our discussion of the set of real numbers, we found that every real number has an opposite which is called its ADDITIVE INVERSE. The sum of a real number and its additive inverse is always zero. For example, 2 and -2 are additive inverses of each other since $2 + (-2) = 0$.

Similarly, every polynomial has an additive inverse. The sum of a polynomial and its additive inverse is zero.

Suppose we wished to find the additive inverse of the polynomial $(x - 3y + 2z)$. We must look for a polynomial which when added to $(x - 3y + 2z)$ will equal zero. That is, we must find a polynomial that satisfies the condition

$$(x - 3y + 2z) + \text{▨} = 0 \cdot$$

From our rules for adding polynomials, we know that

$$\begin{aligned} & (x - 3y + 2z) + (-x + 3y - 2z) \\ = & \ x - 3y + 2z - x + 3y - 2z \\ = & \ (x - x) + (-3y + 3y) + (2z - 2z) \\ = & \ 0 \end{aligned}$$

Thus, the additive inverse of $(x - 3y + 2z)$ must be $(-x + 3y - 2z)$.

Note that this additive inverse could have been obtained by taking the additive inverse of every term in the original polynomial $(x - 3y + 2z)$. The additive inverse of x is $-x$, of $-3y$ is $3y$, and of $2z$ is $-2z$. Thus, the additive inverse of $(x - 3y + 2z)$ is $(-x + 3y - 2z)$.

> In general, the additive inverse of any polynomial can be determined by taking the additive inverse of each term in the sum.

Thus,

1. The additive inverse of $(-x^2 + 7x)$ is $(x^2 - 7x)$ since $(-x^2 + 7x) + (x^2 - 7x) = (-x^2 + x^2) + (7x - 7x) = 0$.

2. The additive inverse of $(-a - b)$ is $(a + b)$ since $(-a - b) + (a + b) = (-a + a) + (-b + b) = 0$.

Self-correcting Exercise #5

Answers may be found on page 57 of this lesson.

1. State the additive inverse of each of the following polynomials.

(a) x

____

(b) $a + b$

____

(c) -5

____

(d) $-a + b$

____

(e) $x - y$

____

(f) $-x - 5$

____

(g) $a^2 - 2ab + b^2$

____

(h) $-xy - y^2 + 2y + 5$

____

(i) $mn - n - m$

____

(j) $-a^2 + b^2 - c^2 - ab + bc + 1$

____

C. Subtraction of Polynomials

Recall that in order to subtract one real number from another, you must add the additive inverse of the subtrahend to the minuend. For example, the subtraction question 4 - 7 can be changed to the addition question 4 + (-7) and the rules for adding real numbers can then be applied.

$$\text{i.e.} \quad 4 - 7 = 4 + (-7) = -3$$

(Operation is changed to addition.)

(Additive inverse of 7 is -7.)

Subtraction of polynomials can be defined in a similar manner. In order to subtract one polynomial from another, you must add the additive inverse of the second polynomial to the first polynomial. Once the subtraction question has been changed to an addition question, the rules for adding polynomials can then be applied. The difference of two polynomials is always a polynomial.

EXAMPLE: Subtract the polynomial $(-5a + 2b - 3c)$ from the polynomial $(-8a + 3b - 6c)$.

Solution

$$(-8a + 3b - 6c) - (-5a + 2b - 3c)$$

Change the operation to addition and give the additive inverse of the second polynomial. Note that the additive inverse of $-5a + 2b - 3c$ is $5a - 2b + 3c$.

$$= (-8a + 3b - 6c) + \underbrace{(5a - 2b + 3c)}$$

(Operation is changed to addition.)

(This is the additive inverse of $-5a + 2b - 3c$.)

Remove brackets and write as a single polynomial.

$$= -8a + 3b - 6c + 5a - 2b + 3c$$

Group like terms together.

$$= (-8a + 5a) + (3b - 2b) + (-6c + 3c)$$

$$= \underline{\underline{-3a + b - 3c}}$$

(Note that the difference is also a polynomial.)

We can also use the vertical method to subtract polynomials. The polynomial that is to be subtracted is placed beneath the other polynomial. Like terms are placed under each other. In order to subtract, we must take the additive inverse of every term in the lower polynomial. This can be done by changing the signs of all the terms in this polynomial. (This step need only be done mentally.) Then the rules for adding like terms can be applied.

EXAMPLE: Subtract $(2x^2 + x - 6)$ from $(3x^2 + 4x + 1)$

Solution

Place the polynomial that is to be subtracted below the other polynomial, lining up like terms.

$$\begin{array}{r}3x^2 + 4x + 1 \\ \underline{2x^2 + x - 6} \end{array} \leftarrow \text{THIS IS THE POLYNOMIAL THAT IS TO BE SUBTRACTED}$$

Mentally change the sign of each term in the lower polynomial and then add like terms.

$$\begin{array}{|r|r|r|} 3x^2 & + 4x & + 1 \\ 2x^2 & + x & - 6 \\ \hline x^2 & + 3x & + 7 \end{array}$$

Additive inverse of $2x^2$ is $-2x^2$. Sum of $3x^2$ and $-2x^2$ is x^2.

Additive inverse of -6 is 6. Sum of 1 and 6 is 7.

Additive inverse of x is -x. Sum of 4x and -x is 3x.

In each case, subtract the lower polynomial from the one above it.

1. $\begin{array}{r} 2a + b - c \\ \underline{a - b + 3c} \\ \underline{\quad} + 2b - \underline{\quad} \end{array}$

2. $\begin{array}{r} 3a + 2b - 4 \\ \underline{-3a + 4b } \\ \underline{\quad} - \underline{\quad} - 4 \end{array}$

3. $\begin{array}{r} 2x - y + 3 \\ \underline{- 2y + 5} \\ 2x + \underline{\quad} - \underline{\quad} \end{array}$

Self-correcting Exercise #6

Answers may be found on page 57 of this lesson.

1. Perform the following subtractions.

(a) $(5a - 6) - (3a - 2)$
$= (5a - 6) + (-3a + 2)$
$=$

(b) $(5x + 2y + 3z) - (-x + 2y - 5z)$
$=$

(c) $(3x^2 - 2y^2 - 4xy) - (7y^2 - xy)$
$=$

(d) $(2x - y + 3) - (-2y + 5)$
$=$

2. Subtract the lower polynomial from the one above it. Check your answers by adding.

(a) $\begin{array}{r} 3x + 4 \\ x - 1 \\ \hline 2x + 5 \end{array}$ **Check** $\begin{array}{r} x - 1 \\ 2x + 5 \\ \hline 3x + 4 \end{array}$

CHANGE SIGNS → OF BOTH THESE TERMS AND THEN ADD.

SAME

(b) $\begin{array}{r} 3x^2 - 2x - 1 \\ 5x + 6 \\ \hline \end{array}$ **Check**

(c) $\begin{array}{r} 1 - t + t^2 \\ - t + 2t^2 \\ \hline \end{array}$ **Check**

(d) $\begin{array}{r} 2a - b + 3c \\ a + b - 4c \\ \hline \end{array}$ **Check**

3. Simplify the following expressions.

(a) $(3a + 2b) + (5a - 6b) - (2a - 4b)$
$=$

(b) $(2x - y) - (3x + 2y) - (x - y)$
$=$

(c) $(x^2 + 3xy - 4y^2) - (3x^2 - 4xy) - (-2x^2 + 3xy - 4y^2)$
=

D. Multiplication of Polynomials

The operation of finding the product of two polynomials becomes more involved as the number of terms in the polynomials increases.

The product of two monomials can be determined by grouping numerical factors together and factors with like bases together. Review page 33 of lesson C where products of this type are handled.

EXAMPLE: Find the product of $-3xy^2$ and $5x^3y$.

Solution

$$\begin{aligned} &(-3xy^2)(5x^3y) \\ &= (-3 \times 5)(x \times x^3)(y^2 \times y) \\ &= -15 \times x^{1+3} \times y^{2+1} \\ &= \underline{\underline{-15x^4y^3}} \end{aligned}$$

Find the following products.

1. $-2a^2 \times 3ab = (-2 \times 3)(a^2 \times \_\_\_)b =$

2. $-5st \times -3s^2t^2 = (\_\_\_ \times \_\_\_)(s \times \_\_\_)(\_\_\_ \times t^2) =$

3. $4xy \times 2xy^2z =$

The product of a monomial and a polynomial with any number of terms can be determined by applying the distributive property. The monomial is considered to be the multiplier and is distributed over every term in the polynomial.

EXAMPLE: Find the product of $-7y^2$ and $(-3y^3 + 7y - 1)$.

Solution

The monomial $-7y^2$ must be distributed over the three terms $-3y^3$, $7y$, and -1 in the polynomial $(-3y^3 + 7y - 1)$.

$$\begin{aligned} &(-7y^2)(-3y^3 + 7y - 1) \\ &= (-7y^2)(-3y^3) + (-7y^2)(7y) + (-7y^2)(-1) \end{aligned}$$

Simplify each term by finding the product of the monomials involved.

$$= \underline{\underline{21y^5 - 49y^3 + 7y^2}}$$

In general, to multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and find the sum of these individual products.

Find the following products.

1. $-3(a - b + c) = (-3)(\_\_\_) + (-3)(-b) + (-3)(\_\_\_) = \_\_\_ + \_\_\_ - \_\_\_$

2. $4x(x + y) = 4x(\_\_\_) + 4x(\_\_\_) = \_\_\_ + \_\_\_$

3. $-2b(b^2 - 3bc + c^2) = (-2b)(\_\_\_) + (-2b)(\_\_\_) + (\_\_\_)(\_\_\_) = \_\_\_\_\_\_\_\_$

The product of two binomials can also be determined by applying the distributive property. The first binomial is considered to be the multiplier and is distributed over the two terms in the second binomial.

EXAMPLES: Find the product of $(2xy + 5)$ and $(4x - 5y^2)$.

Solution

The binomial $(2xy + 5)$ must be distributed over the two terms $4x$ and $-5y^2$ in the binomial $(4x - 5y^2)$.

$$\begin{aligned} &(2xy + 5)(4x - 5y^2) \\ =\ &(2xy + 5)(4x) + (2xy + 5)(-5y^2) \end{aligned}$$

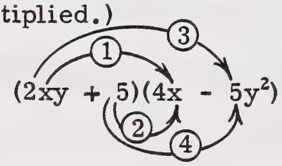
Use the distributive property again to find the products $(2xy + 5)(4x)$ and $(2xy + 5)(-5y^2)$.

$$= (2xy)(4x) + (5)(4x) + (2xy)(-5y^2) + (5)(-5y^2)$$

Simplify each term by finding the product of the monomials involved.

$$= \underline{\underline{8x^2y + 20x - 10xy^3 - 25y^2}}$$

In multiplying two binomials, note that four separate products are involved. The products can be written down immediately by using the following procedure. (Arrows show the order in which terms can be multiplied.)



$$(2xy + 5)(4x - 5y^2) = \overbrace{(2xy)(4x)}^{\textcircled{1}} + \overbrace{5(4x)}^{\textcircled{2}} + \overbrace{(2xy)(-5y^2)}^{\textcircled{3}} + \overbrace{5(-5y^2)}^{\textcircled{4}}$$

$$= \underline{\underline{8x^2y + 20x - 10xy^3 - 25y^2}}$$

In general, to multiply two binomials, multiply each term of one binomial by each term of the other and add the four products.

Find the following products. (Combine any like terms in each product.)

1. $(3m + 5n)(2m - n) = (3m)(2m) + (5n)(2m) + (3m)(-n) + (5n)(-n)$

$= \underline{\qquad\quad} + \underline{\qquad\quad} - \underline{\qquad\quad} - \underline{\qquad\quad}$

$= \underline{\qquad\quad} + 7mn - \underline{\qquad\quad}$

2. $(-3x + 4)(5x + 7) = (-3x)(5x) + (4)(\underline{\quad}) + (-3x)(7) + (\underline{\quad})(7)$

$= \underline{\qquad\quad} + \underline{\qquad\quad} - \underline{\qquad\quad} + \underline{\qquad\quad}$

$= \underline{\qquad\quad} - \underline{\qquad\quad} + \underline{\qquad\quad}$

3. $(a^2 - b)(a + 2b) = (a^2)(\underline{\quad}) + (-b)(\underline{\quad}) + (\underline{\quad})(2b) + (-b)(\underline{\quad})$

$= \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$

4. $(3z - 5)(2z - 3) = (\underline{\quad})(\underline{\quad}) + (\underline{\quad})(\underline{\quad}) + (\underline{\quad})(\underline{\quad}) + (\underline{\quad})(\underline{\quad})$

$= \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$

$= \underline{\qquad\qquad\qquad\qquad\qquad\qquad}$

This method for multiplying two binomials can be extended to cover the product of any two polynomials. In finding such a product, every term of the one polynomial must be multiplied by every term of the second polynomial. It is most convenient to find products of this type by arranging your work in vertical, rather than horizontal, form.

The polynomial with the smaller number of terms is placed beneath the polynomial with the larger number of terms. All the individual products are determined, placing any like terms under each other, and then the partial products are added together. The product of any two polynomials is always a polynomial.

EXAMPLE: Find the product of $(2x^2 + 3x - 5)$ and $(-4x + 3)$.

Solution

$$\begin{array}{rrrr}
 & 2x^2 + & 3x & - \ 5 \\
 & & -4x & + \ 3 \\
\hline
-8x^3 & - \ 12x^2 & + \ 20x & \\
 & 6x^2 & + \ 9x & - \ 15 \\
\hline
-8x^3 & - \ 6x^2 & + \ 29x & - \ 15
\end{array}$$

These are the products obtained by multiplying every term in $(2x^2 + 3x - 5)$ by $-4x$.

These are the products obtained by multiplying every term in $(2x^2 + 3x - 5)$ by 3.

In the final step, you add the six products together.

Note that the product $-8x^3 - 6x^2 + 29x - 15$ is also a polynomial.

Find the following products.

1.
$$\begin{array}{rrrr}
 & 4x^2 & - \ 20x & + \ 25 \\
 & & 2x & - \ 5 \\
\hline
8\_\_ & - \ 40\_\_ & + \ \_\_x & \\
 & - \ 20\_\_ & + \ \_\_x & - \ \_\_ \\
\hline
\_\_x^3 & - \ \_\_x^2 & + \ \_\_x & - \ \_\_
\end{array}$$

2.
$$\begin{array}{llllll}
a & - \ b & + \ c & & & \\
a & - \ b & + \ c & & & \\
\hline
a^2 & - \ \_\_ & + \ ac & & & \\
 & - \ ab & & + \ \_\_^2 & - \ bc & \\
 & & + \ \_\_ & & - \ \_\_ & + \ \_\_^2 \\
\hline
\end{array}$$

Self-correcting Exercise #7

Answers may be found on page 58 of this lesson.

1. Find the following products. (Collect any like terms in each product.)

(a) $-5a^2b^3c \times 7ac^5$

=

(b) $4x^2 \times 3xy^3$

=

(c) $2a(a^2 - ab + b^2)$

=

(d) $-3x(y^2 + y - 7)$

=

(e) $-1(x^2 - y^2)$

=

(f) $(2a + b)(3a^2 + 5)$

=

(g) $(x - y)(x + 5)$
$=$

(h) $(2x + 5)(3x - 2)$
$=$

(i) $(x^2 + 3x - 4)(-x + 5)$
Use vertical form.

(j) $(-4a + 3b - 2c)(a - b - c)$
Use vertical form.

E. Division of Polynomials

In the set of polynomials, the operations of addition, subtraction, and multiplication are CLOSED. This means that the sum, difference, or product of two polynomials is always a polynomial. But, the operation of division is not closed in this set. Division is not possible when the divisor is equal to zero. Also, a polynomial is only obtained when the divisor is a factor of the dividend.

The quotient of two monomials can be written in simplest form by grouping numerical factors together and grouping factors with like bases together. Then, the laws of exponents can be applied. Review pages 34 and 35 of lesson C where quotients of this type are handled.

EXAMPLE 1: Divide $15a^2x$ by $3ax$.
Solution

$$\frac{15a^2x}{3ax} = \frac{15}{3} \times \frac{a^2}{a} \times \frac{x}{x} = 5 \times a^{2-1} \times x^{1-1} = 5ax^0 = \underline{\underline{5a}}$$

Note that in this case the quotient, $5a$, is a polynomial. ($3ax$ is a factor of $15a^2x$ since $3ax \times 5a = 15a^2x$.)

EXAMPLE 2: Divide $12ab^5c^3$ by $-8b^6c^2$.
Solution

$$\frac{12ab^5c^3}{-8b^6c^2} = \frac{12}{-8} \times \frac{a}{1} \times \frac{b^5}{b^6} \times \frac{c^3}{c^2} = \frac{3}{-2} \times \frac{a}{1} \times \frac{1}{b^{6-5}} \times \frac{c^{3-2}}{1} = \boxed{\frac{3ac}{-2b}}$$

Note that in this case the quotient, $\frac{3ac}{-2b}$, is not a polynomial since the variable term $-2b$ appears in the denominator. ($-8b^6c^2$ is not a factor of $12ab^5c^3$.)

Simplify the following quotients. Then decide whether or not the quotient is a polynomial.

1. $\frac{18x^3}{3x} = \frac{18}{3} \times \frac{x^3}{x} =$ ______

   Is the quotient a polynomial? __________

2. $\frac{-8t^2}{2t^3} = \frac{-8}{2} \times \frac{\quad}{\quad} = \frac{\quad}{\quad}$

   Is the quotient a polynomial? __________

3. $\frac{-42a^2b^2}{-7a^3b} = \frac{-42}{-7} \times \frac{\quad}{\quad} \times \frac{\quad}{\quad} = \frac{\quad}{\quad}$

   Is the quotient a polynomial? __________

4. $\frac{36x^3y^2z}{8x^2y} = \frac{36}{8} \times \frac{\quad}{\quad} \times \frac{\quad}{\quad} \times \frac{\quad}{\quad} = \frac{\quad}{\quad}$

   Is the quotient a polynomial? __________

The quotient of a polynomial and a monomial can be written in simplest form by expressing the quotient as a product and then applying the distributive property.

EXAMPLE 1: Divide $(6x^4 - 9x^3 + 3x^2)$ by $-3x$.

Solution

$$\begin{aligned}
\frac{6x^4 - 9x^3 + 3x^2}{-3x} &= \frac{1}{-3x} \times (6x^4 - 9x^3 + 3x^2) \\
&= \frac{1}{-3x}(6x^4) + \frac{1}{-3x}(-9x^3) + \frac{1}{-3x}(3x^2) \quad \left\{\begin{array}{l}\text{Distributive}\\ \text{Property}\end{array}\right. \\
&= \frac{6x^4}{-3x} + \frac{-9x^3}{-3x} + \frac{3x^2}{-3x} \\
&= \underline{\underline{-2x^3 + 3x^2 - x}}
\end{aligned}$$

(In doing questions of this type, you could begin at this step where you divide every term in the polynomial by the divisor $-3x$. Just keep in mind that the distributive property allows you to do this.)

Note that in this case the quotient, $-2x^3 + 3x^2 - x$, is a polynomial. ($-3x$ is a factor of every term in the polynomial $6x^4 - 9x^3 + 3x^2$.)

EXAMPLE 2: Divide $(x^2 - x^3y)$ by xy.

Solution

$$\frac{x^2 - x^3y}{xy} = \frac{x^2}{xy} + \frac{-x^3y}{xy} = \boxed{\frac{x}{y} - x^2}$$

Note that in this case the quotient, $\frac{x}{y} - x^2$, is not a polynomial since the variable term y appears as a denominator. (xy is not a factor of each of the terms in the polynomial $x^2 - x^3y$.)

Find the following quotients. Then decide whether or not each quotient is a polynomial.

1. $\frac{6x^2 + 9x}{3x} = \frac{\quad}{3x} + \frac{\quad}{3x} = \underline{\qquad\qquad}$

   Is the quotient a polynomial? ____________

2. $\frac{25y^5 - 15y^3 + 10y^2 - 5y}{5y} = \frac{\quad}{5y} + \frac{-15y^3}{5y} + \frac{\quad}{5y} + \frac{\quad}{5y} = \underline{\qquad\qquad\qquad\qquad}$

   Is the quotient a polynomial? ____________

3. $\frac{3y^2 + 6y^3 - 2y - 1}{3y^2} = \frac{\quad}{3y^2} + \frac{\quad}{3y^2} + \frac{\quad}{3y^2} + \frac{\quad}{3y^2} = \underline{\quad} + \underline{\quad} - \frac{2}{3y} - \underline{\qquad}$

   Is the quotient a polynomial? ____________

The quotient of two polynomials can be written in simplest form by employing a long division process similar to the process used in dividing large integers. In arithmetic, you learned to find quotients as follows:

$$\begin{array}{r} 13 \\ 21\overline{)273} \\ \underline{21} \\ 63 \\ \underline{63} \\ 0 \end{array} \qquad\qquad \begin{array}{r} 26 \\ 103\overline{)2765} \\ \underline{206} \\ 705 \\ \underline{618} \\ 87 \end{array}$$

Therefore, $\frac{273}{21} = 13$ Therefore, $\frac{2765}{103} = 26\frac{87}{103}$

Now, study the following examples and note how the long division process can be used to find the quotient of two polynomials.

EXAMPLE 1: Divide $(2x^2 + x - 6)$ by $(x + 2)$.
Solution

Step 1: Arrange the divisor $(x + 2)$ and the dividend $(2x^2 + x - 6)$ as in a long division question. Both polynomials should be written in descending powers of x.	$\underset{\text{divisor}}{\underset{\uparrow}{x+2}}\;\overline{)\,2x^2+\underset{\text{dividend}}{\underset{\nwarrow}{x}}-6}$
Step 2: Divide x into $2x^2$ and insert the quotient 2x above the division symbol. Multiply the quotient 2x by the divisor $(x + 2)$ and place the terms of this product below similar terms in the dividend.	$\begin{array}{rl} & \;2x \\ x+2 & \overline{)\,2x^2 + x - 6} \\ & \;\;2x^2+4x \end{array}$
Step 3: Subtract the polynomial $(2x^2 + 4x)$ from the first two terms in the dividend.	$\begin{array}{rl} & \;2x \\ x+2 & \overline{)\,2x^2 + x - 6} \\ & \;\;\underline{2x^2+4x} \\ & \qquad\quad -3x \end{array}$
Step 4: Divide x into $-3x$ and insert the quotient -3 beside the first quotient, 2x. Multiply the quotient -3 by the divisor $(x + 2)$ and place the terms of this product below similar terms in the dividend. (Rewrite the constant term -6 so it is beside the term $-3x$.)	$\begin{array}{rl} & \;2x - 3 \\ x+2 & \overline{)\,2x^2 + x - 6} \\ & \;\;\underline{2x^2+4x} \\ & \qquad\quad -3x-6 \\ & \qquad\quad -3x-6 \end{array}$
Step 5: Subtract the polynomial $(-3x - 6)$ from the polynomial $(-3x - 6)$. Division is complete since the remainder is zero.	$\begin{array}{rl} & \qquad\;\swarrow\text{Quotient} \\ & \;2x - 3 \\ x+2 & \overline{)\,2x^2 + x - 6} \\ & \;\;\underline{2x^2+4x} \\ & \qquad\quad -3x-6 \\ & \qquad\quad \underline{-3x-6} \\ & \qquad\qquad\quad 0 \end{array}$

Therefore, $\dfrac{2x^2 + x - 6}{x + 2} = 2x - 3$

Note that in this case the quotient, $(2x - 3)$, is a polynomial. $(x + 2)$ is a factor of $(2x^2 + x - 6)$ since $(x + 2)(2x - 3) = 2x^2 + x - 6$.

EXAMPLE 2: Divide $(3y^3 - 77y + 13)$ by $(y + 5)$.

Solution

Note that the polynomial $3y^3 - 77y + 13$ is written in descending powers of y, but does not contain any y^2 term. You should insert the term $0y^2$ to hold the place of the y^2 term.

$$\begin{array}{r|rrrrl}
\multicolumn{1}{r}{} & 3y^2 & -\ 15y & -\ 2 & & \longleftarrow\text{Quotient}\\ \cline{2-5}
y+5 & 3y^3\ + & 0y^2 & -\ 77y & +\ 13 & \\
\multicolumn{1}{r}{} & 3y^3\ + & 15y^2 & & & \\ \cline{2-3}
\multicolumn{1}{r}{} & & -\ 15y^2 & -\ 77y & & \\
\multicolumn{1}{r}{} & & -\ 15y^2 & -\ 75y & & \\ \cline{3-4}
\multicolumn{1}{r}{} & & & -\ 2y & +\ 13 & \\
\multicolumn{1}{r}{} & & & -\ 2y & -\ 10 & \\ \cline{4-5}
\multicolumn{1}{r}{} & & & & 23 & \longleftarrow\text{Remainder}
\end{array}$$

Therefore, $\dfrac{3y^3 - 77y + 13}{y + 5} = (3y^2 - 15y - 2) + \dfrac{23}{y + 5}$.

Note that in this case the quotient, $(3y^2 - 15y - 2) + \dfrac{23}{y + 5}$, is not a polynomial since the variable expression y + 5 appears in the denominator. This means that (y + 5) is not a factor of $3y^3 - 77y + 13$.

Find the following quotients. Then decide whether or not each quotient is a polynomial.

$$\begin{array}{r|rrr}
\multicolumn{1}{r}{} & x & & \\ \cline{2-4}
2x-1 & 2x^2 & +\ 5x & +\ 2\\
\multicolumn{1}{r}{} & & - & \\ \cline{2-3}
\multicolumn{1}{r}{} & & 6x & +\ 2\\
\multicolumn{1}{r}{} & & - & \\ \cline{3-4}
\end{array}$$

Is the quotient a polynomial? ____________

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & x^2 & & & \\ \cline{2-5}
x+2 & x^3 & +\ 0x^2 & -\ 2x & +\ 4\\
\multicolumn{1}{r}{} & & + & & \\ \cline{2-3}
\multicolumn{1}{r}{} & & -2x^2 & - & \\
\multicolumn{1}{r}{} & & & -\ 4x & \\ \cline{3-4}
\multicolumn{1}{r}{} & & & 2x & +\\
\multicolumn{1}{r}{} & & & 2x & +\\ \cline{4-5}
\end{array}$$

Is the quotient a polynomial? ____________

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & x^2 & & & \\ \cline{2-5}
2x-1 & 2x^3 & -\ 7x^2 & +\ 9x & -\ 3\\
\multicolumn{1}{r}{} & & -\ x^2 & & \\ \cline{2-3}
\multicolumn{1}{r}{} & & & +\ 9x & \\
\multicolumn{1}{r}{} & & -\ 6x^2 & + & \\ \cline{3-4}
\multicolumn{1}{r}{} & & & 6x & -\ 3\\
\multicolumn{1}{r}{} & & & & \\ \cline{4-5}
\end{array}$$

Is the quotient a polynomial?

____

Self-correcting Exercise #8

Answers may be found on page 58 of this lesson.

1. Write the following quotients in simplest form.

(a) $\frac{-27x^3y^3}{9xy} =$

(b) $\frac{6a^2}{9b^3} =$

(c) $\frac{2mn^3}{4n^2} =$

(d) $\frac{12x - 15y}{3} =$

(e) $\frac{8n - n^2}{n} =$

(f) $\frac{r^2 - 7r}{r^3} =$

(g) $\frac{8x^3 - 4x^2 - 2x}{-x}$

$=$

(h) $\frac{24a^2b + 6ab^2}{3a^2b^2}$

$=$

2. Use long division to find the following quotients.

(a) $\frac{3x^3 + 23x^2 - 14x + 2}{3x - 1}$

(b) $\frac{x^3 - 2x + 1}{x + 1}$

EXERCISE - Operating With Polynomials

1. Fill in the blanks.

(a) In order to subtract the polynomial p_2 from the polynomial p_1, we must ________________ the ________________ inverse of p_2 to p_1.

(b) When adding two polynomials, we group ________________ terms and then use the ________________ property to add the numerical coefficients of these terms.

(c) The set of all polynomials is closed under the operations of ____________, ________________, and ________________________.

This set is not closed under the operation of ________________.

(d) The quotient of $12a^2 - 15ab$ and $-3a$ is equal to ________________.

(e) The additive inverse of a polynomial is determined by taking the ________________ ____________________ of each term.

(f) The sum of a polynomial and its additive inverse is ____________.

(g) The ________________ property can be used to find the product $2x(-y + z + 5)$.

(h) The additive inverse of the polynomial $x - y - 2z$ is ____________.

(i) When using long division to find the quotient of two polynomials in one variable, both polynomials must be arranged in ______________________ powers of the variable.

(j) The product of the monomial $3b$ and the binomial $(6a - 5b)$ is equal to ____________.

2. Express each of the following in the form 1 - P where P is a polynomial.

EXAMPLE: $1 - a + b$ is equivalent to $1 - (a - b)$.
Check: $1 - (a - b) = 1 + (-a + b) = 1 - a + b$

(a) $1 - x - 3y$

$= 1-(x+3y)$

(b) $1 + m - 5n$

$= 1-(-m+5n)$

(c) $1 - a - 3b - 2c$

= ____________________

(d) $1 + xy + x$

= ____________________

(e) $1 - 4ab + 2ac - 3ad$

= ____________________

(f) $1 + p - q + r$

= ____________________

3. Perform the following operations. Answers should be expressed in simplest form.

(a) $2x - (-2x) =$

(b) $(x^2 + 3x) + (4x^2 - 9x + 7) =$

(c) $(6x^2 - 3x + 2) - (7x^2 + 3x - 5)$
=

(d) $(x^2 + 6xy - 5y^2) + (3x^2 - 9xy + 6y^2) + (-2x^2 - y^2)$
=

(e) $(r^2 + r + 6) - (2r^2 - 2r - 5)$
=

(f) $(2x^2 + 3x - 5) + (x^2 - 5x + 7) - (3x^2 + 2x + 1)$
=

(g) $(a - 2b) - (a + b) + (2a - 5b) - (-a - b)$

(h) $(3mn^3t^2)(-m^2t^3)$

(i) $2m(-m^2 + 3mn - n^2)$

(j) $-4xy(2x^3 - 3x^2y^2 + xy^4)$

(k) $(a - 5)(a^2 + 3)$

(l) $(3x - 7)(4x + 1)$

(m) $(3a + b)(2a - 5b)$

(n) $(2y^2 + 3)(4y^2 - 5)$

(o) $\frac{15m^5n^6}{-3m^2n}$

(p) $\frac{5a + b}{a}$

(q) $\frac{2a^3 + 6a^2 - 12a}{2a}$

(r) $\frac{72a^5 - 24ax}{-6ax^4}$

(s) $\frac{4mn + m^2}{-m}$

(t) $\frac{32a^5 - 64a^4b + 24a^3b^2}{-8a^2}$

4. For each pair of polynomials, first add them and then subtract the second from the first. (Use vertical form.)

(a) $2y^3 - 8y^2 + 3y, \quad 2 + 4y + y^2 - y^3$

Addition

$$\begin{array}{r} 2y^3-8y^2+3y \\ -y^3+y^2+4y+2 \\ \hline y^3-7y^2+7y+2 \end{array}$$

(Place like terms under each other.)

Subtraction

$$\begin{array}{r} 2y^3-8y^2+3y \\ -y^3+y^2+4y+2 \\ \hline 3y^3-9y^2-y-2 \end{array}$$

(b) $c^2d + 3c^2d^2 - cd^2, \quad 5c^2d^2 - 4cd^2 + 2c^2d$

Addition

Subtraction

(c) $t^3 - t^2 + 1, \quad -2t^3 + 4t^2 - 5t + 4$

Addition

Subtraction

(d) $2y^4 - 3y^3 + y^2 - 4, \quad 5y^2 + 2y^3 - 3y^4$

Addition

Subtraction

5. Find the following products. (Use vertical form.)

(a)
$$\begin{array}{r} x^2 - 3x - 5 \\ x + 2 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} -a + b - c \\ a - b \\ \hline \end{array}$$

(c)
$$\begin{array}{r} x^2 - xy + y^2 \\ x + y \\ \hline \end{array}$$

(d)
$$\begin{array}{r} x + y + z \\ -x + y - z \\ \hline \end{array}$$

6. Find the following quotients. (Review pages 34 - 36.)

(a) $2c - 1\overline{)8c^3 - 12c^2 + 6c - 1}$

(b) $5x - 2\overline{)25x^2 - 4}$

(c) $y - 2\overline{)y^3 - 6y + 7}$

(d) $4a + 3b\overline{)28a^2 + 41ab + 15b^2}$

7. Simplify.

(a) $-4(2x^2 - x) + 3x(2 + 4x)$

$=$

(b) $x(2y + 4) - 3y(5 - 2x)$

$=$

Topic Three: Special Products

On pages 28 and 29 of this lesson, you were shown the procedure for finding the product of two binomials. For example, the product of the binomials $(2y + 3)$ and $(3x - 5)$ can be determined by multiplying each term in one binomial by each term in the other binomial and then finding the sum of these four separate products.

$$\text{i.e.,}\quad \begin{aligned}(2y+3)(3x-5) &= (2y)(3x)+(3)(3x)+(2y)(-5)+(3)(-5)\\ &= 6xy+9x-10y-15\end{aligned}$$

Note that the product of these two binomials is a polynomial with four different terms.

When finding the product of two binomials, you will sometimes find that the middle two terms in the product will be like terms. When these like terms are combined, the product will be a binomial or a trinomial rather than a polynomial with four terms. In this section, we will focus our attention on these special binomial products and learn shortcut methods of finding these products.

A. Product of Two Binomials Which Results in a Trinomial

The product of two binomials sometimes results in a trinomial. For example,

$$\begin{array}{lll|l|ll}
\cline{4-4}
1. & (4a+3)(2a+7) & = 8a^2 & +\ 6a + 28a & +\ 21 & = 8a^2 + 34a + 21 \\
2. & (2x^2+1)(3x^2-1) & = 6x^4 & +\ 3x^2 - 2x^2 & -\ 1 & = 6x^4 + x^2 - 1 \\
3. & (5x-2y)(2x-3y) & = 10x^2 & -\ 4xy - 15xy & +\ 6y^2 & = 10x^2 - 19xy + 6y^2 \\
\cline{4-4}
\end{array}$$

In each of these examples, the two middle terms are like terms and can be combined. As a result, the final product has only three terms rather than four. Note that in each of these pairs of binomial factors, the first terms of the factors are like terms and the second terms of the factors are like terms.

1. In the factors $(4a + 3)$ and $(2a + 7)$, the first terms, which are $4a$ and $2a$, are like terms. The second terms, which are 3 and 7, are both constants.
2. In the factors $(2x^2 + 1)$ and $(3x^2 - 1)$, the first terms, which are $2x^2$ and $3x^2$, are like terms and the second terms, which are 1 and -1 are both constants.
3. In the factors $(5x - 2y)$ and $(2x - 3y)$, the first terms, which are $5x$ and $2x$, are like terms and the second terms, which are $-2y$ and $-3y$ are like terms.

In general, if two binomials have like first terms and like second terms, the middle two terms in their product will be like terms. These middle terms can be calculated mentally and only their sum written down. Thus, in calculating products of this type, you can speed up your work by combining like terms mentally and writing down the three terms in the product immediately.

EXAMPLE: Find the product (3x - 4)(2x + 5).

Solution

Note that 3x and 2x are like terms and -4 and 5 are both constants. Therefore, the product will be a trinomial.

$$\left.(3x - 4)(2x + 5) = 6x^2 + \begin{matrix}-8x\\ 15x\end{matrix} - 20\ \right\}\ \text{Mental Step}$$

$$\left.= \underline{\underline{6x^2 + 7x - 20}}\ \right\}\ \text{Write this down.}$$

The arrows above indicate the inner and outer products which are calculated mentally and then added together.

Fill in the blanks in the chart below in order to find the following products.

Factors	Mental Calculations			Trinomial Product
	First Term	Second Term	Third Term	
$(2a - 5)(3a + 4)$	$(2a)(3a) = 6a^2$	$(-5)(3a) + (2a)(4)$ $= -15a + 8a$ $= -7a$	$(-5)(4)$ $= -20$	$6a^2 - 7a - 20$
$(x + 4y)(3x - 5y)$	$(x)(3x) =$	$(4y)(3x) + (x)(-5y)$ $=$	$(4y)(-5y)$ $=$	
$(x - 4)(x + 7)$		$(-4)(x) + (x)(7)$ $=$		
$(-x + 3)(2x - 4)$	$(-x)(2x) =$			
$(4a + 2b)(2a - 3b)$			$(2b)(-3b)$ $=$	
$(6x - 1)(3x - 2)$				

Now, write down the three terms in each product below.

1. $(2x + 5)(3x - 8) =$ ___ - ___ - ___
2. $(3y - x)(2y + 3x) =$ ___ + ___ - ___
3. $(a + 5b)(-2a + 4b) =$ ___ - ___ + ___

Self-correcting Exercise #9

Answers may be found on page 59 of this lesson.

1. Decide whether or not the middle two terms in each of the following products are like terms.

	Yes or No?
(a) $(5x - 7)(2x - 9)$	______
(b) $(3a + 2b)(5a - b)$	______
(c) $(2x + y)(x + 3)$	______
(d) $(a - b)(a^2 + b)$	______
(e) $(2a + 5y)(3a - 7y)$	______
(f) $(x + y)(2x + z)$	______
(g) $(2m - n)(3m + 4n)$	______
(h) $(a^2 - 1)(a^3 + 7)$	______

2. Find the following products. Write down the three terms in each product immediately.

   (a) $(x + 2)(x + 5) = x^2 + 7x +$ ______

   (b) $(a - 3)(a + 1) =$ ______________

   (c) $(y - 5)(y - 9) =$ ______________

   (d) $(-x + 2)(x + 7) =$ ______________

   (e) $(2x - 3)(3x + 5) =$ ______________

(f) $(9a + 2)(-a - 6) = \underline{\qquad\qquad\qquad}$

(g) $(3x - 5y)(5x - y) = \underline{\qquad\qquad\qquad}$

(h) $(2m - 3n)(m + n) = \underline{\qquad\qquad\qquad}$

(i) $(5x^2 + 2)(x^2 - 3) = \underline{\qquad\qquad\qquad}$

3. Simplify the following expressions.

(a) $7x^2 - (2x - 3)(5x + 7) = 7x^2 - (\underline{\quad} - \underline{\quad} - \underline{\quad})$
$= 7x^2 + (\underline{\quad} + \underline{\quad} + \underline{\quad})$
$=$

(b) $(2x + 5)(3x + 1) - (x - 3)(x + 2) = (\underline{\qquad\qquad}) - (\underline{\qquad\qquad})$
$= (\underline{\qquad\qquad}) + (\underline{\qquad\qquad})$
$=$

B. Product of Two Binomials Which Results in a Binomial

The product of two binomials sometimes results in a binomial. For example,

1. $(2a + 3)(2a - 3) = 4a^2 \boxed{+ 6a - 6a} - 9 = 4a^2 - 9$
2. $(2x^2 + 5)(2x^2 - 5) = 4x^4 \boxed{+ 10x^2 - 10x^2} - 25 = 4x^4 - 25$
3. $(5x + 3y)(5x - 3y) = 25x^2 \boxed{+ 15xy - 15xy} - 9y^2 = 25x^2 - 9y^2$

In each of these examples, the middle two terms are like terms whose sum is zero. As a result, the final product has only two terms.

Note that in each of the pairs of binomial factors, the first terms of the factors are identical and the second terms of the factors are additive inverses of each other.

1. In the factors $(2a + 3)$ and $(2a - 3)$, the first terms are both $2a$. The last terms, which are 3 and -3, are additive inverses of each other.
2. In the factors $(2x^2 + 5)$ and $(2x^2 - 5)$, the first terms are both $2x^2$. The last terms, which are 5 and -5, are additive inverses of each other.
3. In the factors $(5x + 3y)$ and $(5x - 3y)$, the first terms are both $5x$. The last terms, which are $3y$ and $-3y$, are additive inverses of each other.

In general, if two binomials have the same first terms, and second terms which are additive inverses of each other, the sum of the middle two terms in their product will be zero. The product can be written as the difference of two squares. Thus, in calculating products of this type, you can speed up your work by squaring the first term of either binomial, squaring the second term of either binomial , and subtracting these two squares.

EXAMPLE: Find the product $(3ab - 7c)(3ab + 7c)$.

Solution

Note that $3ab$ and $3ab$ are identical terms and $-7c$ and $7c$ are additive inverses of each other.

$$(3ab - 7c)(3ab + 7c) = \underset{\nwarrow\ \text{(Square of first term)}}{(3ab)^2} \underset{\overset{\uparrow}{\text{(Subtract)}}}{-} \underset{\nearrow\ \text{(Square of second term)}}{(7c)^2} = 3^2a^2b^2 - 7^2c^2 = \underline{\underline{9a^2b^2 - 49c^2}}$$

Fill in the blanks in the chart below in order to find the following products.

Factors	Square of First Term	Square of Second Term	Binomial Product (Difference of Squares)
$(4x - 3y)(4x + 3y)$	$(4x)^2 = 16x^2$	$(3y)^2 = 9y^2$	$16x^2 - 9y^2$
$(a - 7b)(a + 7b)$		$(7b)^2 = \underline{\qquad}$	$a^2 - \underline{\qquad}$
$(5rs - t)(5rs + t)$	$(5rs)^2 = \underline{\qquad}$		$\underline{\qquad} - t^2$
$(m^2n - 2p)(m^2n + 2p)$	$(m^2n)^2 = m^4n^2$		
$(6ab + 8c^2)(6ab - 8c^2)$			
$(3m^3 + nt)(3m^3 - nt)$			
$\left(\frac{1}{3}y + 4\right)\left(\frac{1}{3}y - 4\right)$			
$\left(\frac{xy}{5} - 1\right)\left(\frac{xy}{5} + 1\right)$	$\left(\frac{xy}{5}\right)^2 = \underline{\qquad}$		

Self-correcting Exercise #10

Answers may be found on page 60 of this lesson.

1. Decide whether or not the sum of the middle two terms in each of the following products equals zero.

Yes or No?

(a) $(3x + 2)(3x + 2)$ ____________

(b) $(2x - 3y)(2x - 3y)$ ____________

(c) $(2m^2 + 5)(2m^2 - 5)$ ____________

(d) $(a - 2b)(a^2 + 2b)$ ____________

(e) $(2m - 3n)(3m - 2n)$ ____________

(f) $(x^2y + z)(x^2y - z)$ ____________

(g) $(x^2 - 7y)(x^2 + 7y)$ ____________

2. Find the following products. Write down the two terms in each product immediately.

(a) $(x + 5)(x - 5) = (x)^2 - (5)^2 =$

(b) $(2x - 9)(2x + 9) =$

(c) $(4y + 7)(4y - 7) =$

(d) $(2xy + 3)(2xy - 3) =$

(e) $(3a + 10b)(3a - 10b) =$

(f) $(-x + 4)(x + 4) = \underbrace{(4 - x)(4 + x)}_{\text{(Put in standard form.)}} =$

(g) $(2x + 7y)(-2x + 7y) =$

(h) $(x^2 - 6)(x^2 + 6) =$

(i) $(3a^2b + 5c)(3a^2b - 5c) =$

(j) $(1 + xyz)(1 - xyz) =$

3. Simplify the following expressions.

(a) $(2x + z)(2x - z) - (z - 5x)(z + 5x)$

=

(b) $(4x - 1)(4x + 1) + (2x - 3)$

=

C. The Square of a Binomial

The square of a binomial represents the product of two identical binomials and always results in a trinomial. For example,

$$\begin{array}{lll} \begin{array}{ll} 1. & (3x + 2)^2 = (3x + 2)(3x + 2) = 9x^2 \\ 2. & (2y^2 - 5)^2 = (2y^2 - 5)(2y^2 - 5) = 4y^4 \\ 3. & (2a + 3b)^2 = (2a + 3b)(2a + 3b) = 4a^2 \end{array} & \boxed{\begin{array}{l} +\ 6x + 6x \\ -\ 10y^2 - 10y^2 \\ +\ 6ab + 6ab \end{array}} & \begin{array}{l} + 4 = 9x^2 + 12x + 4 \\ + 25 = 4y^4 - 20y^2 + 25 \\ + 9b^2 = 4a^2 + 12ab + 9b^2 \end{array} \end{array}$$

In each of these examples, the middle two terms are identical.

Note that in each of these trinomial products, the first term is the square of the first term in the binomial factor. The middle term is twice the product of the two terms in the binomial. The last term is the square of the second term in the binomial factor.

1. $(3x + 2)^2 = (3x)^2 + 2(3x)(2) + (2)^2 = 9x^2 + 12x + 4$
   The first term in the product is the square of 3x, the first term in the binomial. The middle term is twice the product of 3x and 2, the two terms in the binomial. The last term is the square of 2, the second term in the binomial.
2. $(2y^2 - 5)^2 = (2y^2)^2 + 2(2y^2)(-5) + (-5)^2 = 4y^4 - 20y^2 + 25$
   Note that $4y^4$ is the square of $2y^2$, $-20y^2$ is twice the product of $2y^2$ and -5, and 25 is the square of -5.
3. $(2a + 3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2 = 4a^2 + 12ab + 9b^2$
   Note that $4a^2$ is the square of 2a, 12ab is twice the product of 2a and 3b, and $9b^2$ is the square of 3b.

Thus, when squaring a binomial factor you can speed up your work by squaring the first term of the binomial, finding twice the product of the two terms of the binomial, squaring the second term of the binomial and adding these three results.

EXAMPLE: Evaluate $(3ab - 7c)^2$.

Solution Remember that $(3ab - 7c)^2$ represents the product of $(3ab - 7c)$ and $(3ab - 7c)$.

$$(3ab - 7c)^2 = \underbrace{(3ab)^2}_{\text{Square of the first term}} + \underbrace{2(3ab)(-7c)}_{\text{Twice the product of the two terms}} + \underbrace{(-7c)^2}_{\text{Square of second term}} = \underline{\underline{9a^2b^2 - 42abc + 49c^2}}$$

All squares of binomials satisfy the following conditions:

1. The first and last terms of the expansion are perfect squares and hence are positive.

2. The middle term of the expansion is positive if the terms of the binomial being squared have the same sign.

EXAMPLES: $(2a + b)^2 = 4a^2 + 4ab + b^2$

MIDDLE TERM IS POSITIVE.

$(-x - y)^2 = x^2 + 2xy + y^2$

3. The middle term of the expansion is negative if the terms of the binomial being squared have opposite signs.

EXAMPLES: $(3x - 5y)^2 = 9x^2 - 30xy + 25y^2$

MIDDLE TERM IS NEGATIVE.

$(-m + 4n)^2 = m^2 - 8mn + 16n^2$

Fill in the blanks in the following chart in order to square the binomials.

Binomial Squared	Calculations			Expansion
	First Term	Middle Term	Last Term	
$(5y - 4z)^2$	$(5y)^2 = 25y^2$	$2(5y)(-4z)$ $= -40yz$	$(-4z)^2 = 16z^2$	$25y^2 - 40yz + 16z^2$
$(x + 2y)^2$		$2(x)(2y)$ = ____	$(2y)^2$ = ____	
$(-2a + 3b)^2$	$(-2a)^2$ = ____	$2(-2a)(3b)$ = ____		
$(ab - 5c)^2$				
$(3x + 8y)^2$			$(8y)^2$ = ____	

Self-correcting Exercise #11

Answers may be found on page 60 of this lesson.

1. Evaluate the following expressions. Write down the three terms in each product immediately.

   (a) $(-5x + 1)^2 = (-5x)^2 + 2(-5x)(1) + (1)^2 =$

   (b) $(3x - 7y)^2 =$

   (c) $(2 - 9x)^2 =$

   (d) $(2ab + 5)^2 =$

   (e) $(-7x - 3)^2 = (-7x)^2 + 2(-7x)(-3) + (-3)^2 =$

   (f) $(2x^2 + 1)^2 =$

2. Simplify the following expressions.

   (a) $7x^2 - (x - 3)^2 = 7x^2 - (\underline{\hspace{6em}})$
   $=$

   (b) $(a + b)^2 + (a - b)^2 - (-a - b)^2$
   $=$

EXERCISE - Special Products

1. Find the following products using the shortest method possible.

(a) $(a - 4)(a + 4) =$

(b) $(r + 4)(r + 2) =$

(c) $(t - 5)(t - 6) =$

(d) $(x - 7)^2 =$

(e) $(3x + y)(3x - y) =$

(f) $(2x + 1)(x - 2) =$

(g) $(-5x + 7)^2 =$

(h) $(-2x + 1)(x - 2) =$

(i) $(5x - y)(3x - 4y) =$

(j) $(2x + b)^2 =$

(k) $(2 - m^2)(2 + m^2) =$

(l) $(2m^2 + 3)(3m^2 - 5) =$

(m) $(3 - 4a)^2 =$

(n) $(3a - bc^2)(3a + bc^2) =$

(o) $(5x^2 + 7)(2x^2 - 3) =$

(p) $(xy - 4)^2 =$

(q) $(3mn - 2)(3mn + 2) =$

(r) $(5x - 1)(3x - 2) =$

(s) $(2y^2 + 4z)^2 =$

(t) $(2a^2 - b^2)(2a^2 + b^2) =$

(u) $(3m - 7n)(3m + n) =$

(v) $(-4y - 7)^2 =$

2. After like terms have been collected, how many terms will each of these products contain ?

	Number of Terms
(a) $(5x - 7y)(-2x + 4y)$	three
(b) $(3a^2b - c^3)(3a^2b + c^3)$	
(c) $(a + b)(a^2 + b^2)$	
(d) $(3a + 2)^2$	
(e) $(1 - z)(1 - z)$	
(f) $(1 - z)(1 + z)$	
(g) $(1 - 2z)(1 + z)$	
(h) $(1 - 2z^2)(1 + z)$	
(i) $(1 - 2z^2)(1 + 2z^2)$	
(j) $(mx + b)(mx - b)$	
(k) $(mx - b)^2$	
(l) $(x + 1)(y + 5)$	
(m) $(x + 2)(x + 5)$	

3. Simplify the following expressions.

(a) $(x + 3)^2 + (3x - 1)^2$

=

(b) $(x - 1)^2 - (-x + 3)(x + 3)$

$= (\underline{\hspace{3cm}}) - (\underline{\hspace{3cm}})$

$= (\underline{\hspace{3cm}}) + (\underline{\hspace{3cm}})$

=

(c) $-5x(x + 3y)^2$

$=$

(d) $(x - y)(x + y) + (x - 2y)^2$

$=$

(e) $2a(a + 1) - (a - 1)(a + 3)$

$= (2a^2 + \underline{\quad}) - (a^2 + \underline{\quad} - \underline{\quad})$

$= (2a^2 + \underline{\quad}) + (-a^2 - \underline{\quad} + \underline{\quad})$

$=$

(f) $(x + 1) + 2(x - 1) - (x + 3)^2$

$= (x + 1) + (\underline{\quad} - \underline{\quad}) - (\underline{\quad} + \underline{\quad} + \underline{\quad})$

$= (x + 1) + (\underline{\quad} - \underline{\quad}) + (\underline{\qquad\qquad\qquad})$

$=$

(g) $(4a + 2b)(4a - 2b) + (3b - 2a)^2$

$=$

Key to Self-correcting Exercises in Lesson D

Exercise #1, page 2

1. (a) yes (b) no (variable in denominator)

(c) yes (d) yes

(e) no (root of variable) (f) no (negative exponent)

(g) yes (h) no ($\sqrt{-1}$ is not real)

(i) no (variable in denominator) (j) no (fractional exponent)

(k) yes (l) no (variable in denominator)

2. (a) $-5x^2$, $3x$, -9 (b) $\frac{2}{3}xy^2z$ (one term)

(c) y, -7 (d) 5, $-a^2b$, $-b^3$

(e) $\sqrt{2}x$, $\sqrt{3}$, $-7x^2$, $4x^3$ (f) abc, $-2b^2$, c

Exercise #2, page 9

1. (a) (i) The first term is of degree 2, the second term is of degree 3 and the third term is of degree 5.
(ii) The polynomial is of degree 5.

(b) (i) The first term is of degree 2, the second term is of degree 3, the third term is of degree 3, the fourth term is of degree 1, and the fifth term is of degree 0.
(ii) The polynomial is of degree 3.

	No. of Terms	No. of Variables	Degree
2. (a)	4	2	4 ($2x^2y^2$ is of degree 4.)
(b)	1	3	7 ($3 + 3 + 1 = 7$)
(c)	3	2	8 (a^4b^4 is of degree 8.)
(d)	4	3	5 ($4a^2bc^2$ is of degree 5.)
(e)	2	2	6 ($3xy^5$ is of degree 6.)

Exercise #2, continued

3. (a) polynomial over the rationals (The coefficients 1, $\frac{1}{2}$, and 4 all belong to Q.)

   (b) polynomial over the integers (The coefficients 1 and -1 both belong to I.)

   (c) polynomial over the reals (The coefficients 2π and 3 both belong to R.)

   (d) polynomial over the integers (The coefficients 1, 3, and -2 all belong to I.)

   (e) polynomial over the rationals (The coefficients $\frac{1}{5}$, -3, and 7 all belong to Q.)

4. (a) binomial, linear (b) trinomial, cubic

   (c) monomial, constant (d) binomial, quartic

   (e) trinomial, quadratic (f) monomial, linear

   (g) binomial, quadratic (h) binomial, cubic

Exercise #3, page 14

1. $-4x - 2$; linear $(ax + b)$; $a = -4$, $b = -2$

2. $2x^4 - 7x^3 + 1$; quartic $(ax^4 + bx^3 + cx^2 + dx + e)$; $a = 2$, $b = -7$, $c = 0$, $d = 0$, $e = 1$

3. $x^2 - x$; quadratic $(ax^2 + bx + c)$; $a = 1$, $b = -1$, $c = 0$

4. $\frac{-3}{5}x^3 + \frac{1}{3}x + \frac{2}{5}$; cubic $(ax^3 + bx^2 + cx + d)$; $a = \frac{-3}{5}$, $b = 0$, $c = \frac{1}{3}$, $d = \frac{2}{5}$

5. $-8x^2 + 7$; quadratic $(ax^2 + bx + c)$; $a = -8$, $b = 0$, $c = 7$

Exercise #4, page 21

1. (a) $(-2a^2 + 5a^2) + (3a - 7a) + (4 - 9 + 12) = \underline{3a^2 - 4a + 7}$

   (b) $5y^6 + (-3y^5 + y^5) - y^4 + (-2y^2 - 2y^2) + (y + y) = \underline{5y^6 - 2y^5 - y^4 - 4y^2 + 2y}$

   (c) $(a + 2a - 6a - 3a) + (b - 3b + 4b - b) + (c - 2c + 3c + 2c) = \underline{-6a + b + 4c}$

Exercise #4, continued

2. (a)
$$\begin{array}{rrr} 2ay & -\ 3y & +\ 2 \\ 4ay & & -\ 7 \\ -2ay & -\ 6y & +\ 3 \\ \hline 4ay & -\ 9y & -\ 2 \end{array}$$

(b)
$$\begin{array}{rrrr} 4x^5 & +\ 2x^3 & +\ x^2 & -\ 1 \\ & & -\ 8x^2 & +\ 3 \\ -9x^5 & +\ 2x^3 & & +\ 6 \\ \hline -5x^5 & +\ 4x^3 & -\ 7x^2 & +\ 8 \end{array}$$

Exercise #5, page 23

1. (a) $-x$ (b) $-a - b$

(c) 5 (d) $a - b$

(e) $-x + y$ (f) $x + 5$

(g) $-a^2 + 2ab - b^2$ (h) $xy + y^2 - 2y - 5$

(i) $-mn + n + m$ (j) $a^2 - b^2 + c^2 + ab - bc - 1$

Exercise #6, page 26

1. (a) $(5a - 6) + (-3a + 2) = (5a - 3a) + (-6 + 2) = \underline{2a - 4}$

(b) $(5x + 2y + 3z) + (x - 2y + 5z) = (5x + x) + (2y - 2y) + (3z + 5z) = \underline{6x + 8z}$

(c) $(3x^2 - 2y^2 - 4xy) + (-7y^2 + xy) = 3x^2 + (-2y^2 - 7y^2) + (-4xy + xy) = \underline{3x^2 - 9y^2 - 3xy}$

(d) $(2x - y + 3) + (2y - 5) = 2x + (-y + 2y) + (3 - 5) = \underline{2x + y - 2}$

2. (a)
$$\begin{array}{ccc} \underline{\text{Subtract}} & & \underline{\text{Add}} \\ \begin{array}{r} 3x + 4 \\ x - 1 \\ \hline 2x + 5 \end{array} & \xleftrightarrow{\text{SAME}} & \begin{array}{r} x - 1 \\ 2x + 5 \\ \hline 3x + 4 \end{array} \end{array}$$

(b)
$$\begin{array}{ccc} \underline{\text{Subtract}} & & \underline{\text{Add}} \\ \begin{array}{r} 3x^2 - 2x - 1 \\ 5x + 6 \\ \hline 3x^2 - 7x - 7 \end{array} & \xleftrightarrow{\text{SAME}} & \begin{array}{r} 5x + 6 \\ 3x^2 - 7x - 7 \\ \hline 3x^2 - 2x - 1 \end{array} \end{array}$$

(c)
$$\begin{array}{ccc} \underline{\text{Subtract}} & & \underline{\text{Add}} \\ \begin{array}{r} 1 - t + t^2 \\ - t + 2t^2 \\ \hline 1 \quad - t^2 \end{array} & \xleftrightarrow{\text{SAME}} & \begin{array}{r} - t + 2t^2 \\ 1 \quad - t^2 \\ \hline 1 - t + t^2 \end{array} \end{array}$$

(d)
$$\begin{array}{ccc} \underline{\text{Subtract}} & & \underline{\text{Add}} \\ \begin{array}{r} 2a - b + 3c \\ a + b - 4c \\ \hline a - 2b + 7c \end{array} & \xleftrightarrow{\text{SAME}} & \begin{array}{r} a + b - 4c \\ a - 2b + 7c \\ \hline 2a - b + 3c \end{array} \end{array}$$

Exercise #6, continued

3. (a) $(3a + 2b) + (5a - 6b) - (2a - 4b)$
$= -(3a + 2b) + (5a - 6b) + (-2a + 4b)$
$= (3a + 5a - 2a) + (2b - 6b + 4b)$
$= \underline{6a}$

(b) $(2x - y) - (3x + 2y) - (x - y)$
$= (2x - y) + (-3x - 2y) + (-x + y)$
$= (2x - 3x - x) + (-y - 2y + y)$
$= \underline{-2x - 2y}$

(c) $(x^2 + 3xy - 4y^2) - (3x^2 - 4xy) - (-2x^2 + 3xy - 4y^2)$
$= (x^2 + 3xy - 4y^2) + (-3x^2 + 4xy) + (2x^2 - 3xy + 4y^2)$
$= (x^2 - 3x^2 + 2x^2) + (3xy + 4xy - 3xy) + (-4y^2 + 4y^2)$
$= \underline{4xy}$

Exercise #7, page 30

1. (a) $(-5 \times 7)(a^2 \times a)(b^3)(c \times c^5) = \underline{-35a^3b^3c^6}$

(b) $(4 \times 3)(x^2 \times x)(y^3) = \underline{12x^3y^3}$

(c) $2a(a^2) + 2a(-ab) + 2a(b^2) = \underline{2a^3 - 2a^2b + 2ab^2}$

(d) $(-3x)(y^2) + (-3x)(y) + (-3x)(-7) = \underline{-3xy^2 - 3xy + 21x}$

(e) $(-1)(x^2) + (-1)(-y^2) = \underline{-x^2 + y^2}$

(f) $(2a)(3a^2) + (b)(3a^2) + (2a)(5) + (b)(5) = \underline{6a^3 + 3a^2b + 10a + 5b}$

(g) $(x)(x) + (-y)(x) + (x)(5) + (-y)(5) = \underline{x^2 - xy + 5x - 5y}$

(h) $(2x)(3x) + (5)(3x) + (2x)(-2) + (5)(-2) = 6x^2 + 15x - 4x - 10 \Rightarrow \underline{6x^2 + 11x - 10}$

(i)
$$\begin{array}{rrrr}
 & x^2 & +3x & -4 \\
 & & -x & +5 \\
\hline
-x^3 & -3x^2 & +4x & \\
 & +5x^2 & +15x & -20 \\
\hline
-x^3 & +2x^2 & +19x & -20
\end{array}$$

(j)
$$\begin{array}{rrrrrr}
-4a & +3b & -2c & & & \\
a & -b & -c & & & \\
\hline
-4a^2 & +3ab & -2ac & & & \\
 & +4ab & & -3b^2 & +2bc & \\
 & & +4ac & & -3bc & +2c^2 \\
\hline
-4a^2 & +7ab & +2ac & -3b^2 & -bc & +2c^2
\end{array}$$

Exercise #8, page 36

1. (a) $\frac{-27}{9} \times \frac{x^3}{x} \times \frac{y^3}{y} = -3 \times x^2 \times y^2 = \underline{-3x^2y^2}$

(b) $\frac{6}{9} \times a^2 \times \frac{1}{b^3} = \frac{2}{3} \times a^2 \times \frac{1}{b^3} = \boxed{\frac{2a^2}{3b^3}}$

(c) $\frac{2}{4} \times m \times \frac{n^3}{n^2} = \frac{1}{2} \times m \times n = \boxed{\frac{mn}{2}}$

Exercise #8, continued

1. (d) $\frac{12x}{3} + \frac{-15y}{3} = \underline{4x - 5y}$

(e) $\frac{8n}{n} + \frac{-n^2}{n} = \underline{8 - n}$

(f) $\frac{r^2}{r^3} + \frac{-7r}{r^3} = \boxed{\frac{1}{r} - \frac{7}{r^2}}$

(g) $\frac{8x^3}{-x} + \frac{-4x^2}{-x} + \frac{-2x}{-x} = \underline{-8x^2 + 4x + 2}$

(h) $\frac{24a^2b}{3a^2b^2} + \frac{6ab^2}{3a^2b^2} = \boxed{\frac{8}{b} + \frac{2}{a}}$

2. (a)

$$\begin{array}{rrrrr}
 & & x^2 & +8x & -2 \\ \cline{2-5}
3x-1\,\big) & 3x^3 & +23x^2 & -14x & +2 \\
 & 3x^3 & -x^2 & & \\ \cline{2-3}
 & & 24x^2 & -14x & \\
 & & 24x^2 & -8x & \\ \cline{3-4}
 & & & -6x & +2 \\
 & & & -6x & +2 \\ \cline{4-5}
 & & & & 0
\end{array}$$

(b)

$$\begin{array}{rrrrr}
 & & x^2 & -x & -1 \\ \cline{2-5}
x+1\,\big) & x^3 & +0x^2 & -2x & +1 \\
 & x^3 & +x^2 & & \\ \cline{2-3}
 & & -x^2 & -2x & \\
 & & -x^2 & -x & \\ \cline{3-4}
 & & & -x & +1 \\
 & & & -x & -1 \\ \cline{4-5}
 & & & & 2
\end{array}$$

Exercise #9, page 45

1. (a) yes

(b) yes

(c) no; y and 3 are unlike terms.

(d) no; a and a^2 are unlike terms.

(e) yes

(f) no; y and z are unlike terms.

(g) yes

(h) no; a^2 and a^3 are unlike terms.

2. (a) $x^2 + 7x + 10$

(b) $a^2 - 2a - 3$

(c) $y^2 - 14y + 45$

(d) $-x^2 - 5x + 14$

(e) $6x^2 + x - 15$

(f) $-9a^2 - 56a - 12$

(g) $15x^2 - 28xy + 5y^2$

(h) $2m^2 - mn - 3n^2$

(i) $5x^4 - 13x^2 - 6$

Exercise #9, continued

3. (a) $7x^2 - (10x^2 - x - 21) = 7x^2 + (-10x^2 + x + 21) = \underline{-3x^2 + x + 21}$

(b) $(6x^2 + 17x + 5) - (x^2 - x - 6) = (6x^2 + 17x + 5) + (-x^2 + x + 6)$
$= \underline{5x^2 + 18x + 11}$

Exercise #10, page 48

1. (a) no (b) no (c) yes (d) no (e) no (f) yes (g) yes

2. (a) $x^2 - 25$ (b) $4x^2 - 81$ (c) $16y^2 - 49$

(d) $4x^2y^2 - 9$ (e) $9a^2 - 100b^2$ (f) $16 - x^2$

(g) $49y^2 - 4x^2$ (h) $x^4 - 36$ (i) $9a^4b^2 - 25c^2$

(j) $1 - x^2y^2z^2$

3. (a) $(4x^2 - z^2) - (z^2 - 25x^2) = 4x^2 - z^2 - z^2 + 25x^2 = \underline{29x^2 - 2z^2}$

(b) $(16x^2 - 1) + (2x - 3) = \underline{16x^2 + 2x - 4}$

Exercise #11, page 51

1. (a) $25x^2 - 10x + 1$

(b) $(3x)^2 + 2(3x)(-7y) + (-7y)^2 = \underline{9x^2 - 42xy + 49y^2}$

(c) $(2)^2 + 2(2)(-9x) + (-9x)^2 = \underline{4 - 36x + 81x^2}$

(d) $(2ab)^2 + 2(2ab)(5) + (5)^2 = \underline{4a^2b^2 + 20ab + 25}$

(e) $\underline{49x^2 + 42x + 9}$

(f) $(2x^2)^2 + 2(2x^2)(1) + (1)^2 = \underline{4x^4 + 4x^2 + 1}$

2. (a) $7x^2 - (x^2 - 6x + 9) = 7x^2 - x^2 + 6x - 9 = \underline{6x^2 + 6x - 9}$

(b) $(a^2 + 2ab + b^2) + (a^2 - 2ab + b^2) - (a^2 + 2ab + b^2)$
$= a^2 + 2ab + b^2 + a^2 - 2ab + b^2 - a^2 - 2ab - b^2$
$= \underline{a^2 - 2ab + b^2}$

End of Lesson D

Lesson E

FACTORING; RATIONAL EXPRESSIONS

Topic One: Factoring Polynomials

A PRIME NUMBER is a natural number that is divisible only by itself and the number 1. For example, the natural numbers 2, 3, 5, 7, 11, 13,... are all prime numbers. (1 is not considered to be a prime number.)

A COMPOSITE NUMBER is a natural number that has divisors other than itself and 1. Any composite number can be written as the product of primes. When it is written in this manner, it is said to be in FACTORED FORM. For example, note how the composite numbers 24, 45, and 770 can be written as the product of primes.

$$\left.\begin{aligned} 24 &= 2 \times 2 \times 2 \times 3 \\ 45 &= 3 \times 3 \times 5 \\ 770 &= 2 \times 5 \times 7 \times 11 \end{aligned}\right\} \quad \text{All these factors are prime numbers.}$$

In algebra, we are often asked to express a polynomial as a product. Transforming a given polynomial into a product of other polynomials is called POLYNOMIAL FACTORING. When factoring polynomials whose numerical coefficients are integers, we must look for factors which are integers or are polynomials with integral coefficients.

Some polynomials can only be expressed as a product of themselves and 1. A PRIME POLYNOMIAL is a polynomial which cannot be factored into polynomials of lower degree. A polynomial is said to be FACTORED FULLY when it is expressed as the product of prime polynomials.

Factoring a polynomial is the reverse operation of finding the product of polynomials. When finding the product of polynomials, we change an indicated product to an indicated sum. When factoring a polynomial, we change an indicated sum to an indicated product. For example, when evaluating the product $2a^2(a + b)$, we change this product to the sum $2a^3 + 2a^2b$. If we wished to reverse this process, we could change the sum $2a^3 + 2a^2b$ to the product $2a^2(a + b)$. That is, the polynomial $2a^3 + 2a^2b$ could be written in factored form as $2a^2(a + b)$. Note that $2a^3 + 2a^2b$ is a polynomial of degree three and both its factors are of lower degree. $2a^2$ is of degree two and $(a + b)$ is of degree one.

The process of factoring a polynomial involves searching for other polynomials whose product yields the given polynomial. Although this search for polynomial factors involves a fair amount of trial and error work there are some systematic procedures which can be used. In the next part of this lesson, we will discuss some of these procedures.

A. The Highest Common Factor of Variable Terms

The HIGHEST COMMON FACTOR (H.C.F.) of a set of integers is the largest number that will divide evenly into every integer in the set. For example, the H.C.F. of the numbers 16, 24, and 32 is 8 since this is the largest number that will divide evenly into all three of the given numbers.

Check: $\frac{16}{8} = 2, \quad \frac{24}{8} = 3, \quad \frac{32}{8} = 4$ } Note that the quotients 2, 3, and 4 have no factors in common.

Now, suppose you were asked to find the H.C.F. of the variable terms $18x^2$ and $12x^3$. Note that the largest number that will divide evenly into both numerical coefficients is 6. The power of x with the largest exponent that will divide evenly into both literal coefficients is x^2. Thus, the H.C.F. of $18x^2$ and $12x^3$ is $6x^2$.

Check: $\frac{18x^2}{6x^2} = 3, \quad \frac{12x^3}{6x^2} = 2x$ } Note that the quotients 3 and 2x have no factors in common.

When finding the H.C.F. of a set of variable terms, you must go through the following steps.

1. Find the H.C.F. of all the numerical coefficients.
2. Find the H.C.F. of all the literal coefficients.
   (Note: Any variable which occurs in every term will be part of the H.C.F. It will appear in the H.C.F. with an exponent equal to the smallest exponent that appears in all the terms.)
3. Find the product of these numerical and literal factors which you have picked out.
4. Check your work and make sure that this H.C.F. does in fact divide into every term. The quotients you obtain should have no factors in common.

EXAMPLE: Find the H.C.F. of $21x^2y^3z$, $14xy^4$, and $35x^3y^5z^2$.

Solution

1. The H.C.F. of 21, 14, and 35 is 7.
2. Note that the variable x appears in every term. The smallest exponent it has in any term is 1. That is, the H.C.F. of x^2, x, and x^3 is x. The variable y also appears in every term. The smallest exponent it has in any term is 3. That is, the H.C.F. of y^3, y^4, and y^5 is y^3. Since the variable z does not appear in the term $14xy^4$, it is not part of the H.C.F.
3. $\text{H.C.F.} = 7 \times x \times y^3$
   $= 7xy^3$
   Therefore, the largest number which will divide evenly into all three terms is $7xy^3$.

4. Check:

$\frac{21x^2y^3z}{7xy^3} = 3xz, \qquad \frac{14xy^4}{7xy^3} = 2y, \qquad \frac{35x^3y^5z^2}{7xy^3} = 5x^2y^2z^2$

Note that the quotients $3xz$, $2y$, and $5x^2y^2z^2$ have no factors which are common to all three terms.

Self-correcting Exercise #1

Answers may be found on page 63 of this lesson.

1. State the H.C.F. of each of the following sets of terms.

(a) $35, 49, 42$ ____________

(b) a^3, a^2, a^4 ____________

(c) xy^2, x^2y, x^2y^2 ____________

(d) xt, yt, zt ____________

(e) $45x, 72x, 54x$ ____________

(f) $35a^3, 28a^2, 63a^4$ ____________

(g) x^3y^3, x^2y^4, x^2y^3 ____________

(h) $a^2b^2c^2, b^2c, b^2c^3$ ____________

(i) $33m^3, 55n^2, 44m^2n^2$ ____________

2. Fill in the blanks in the following chart.

Variable Terms	Highest Common Numerical Factor	Highest Common x Factor	Highest Common y Factor	Highest Common z Factor	H.C.F.
$27xy^3, \quad 15x^2y^2$	3	x	y^2	———	$3xy^2$
$8x^3, \quad 12x^5$					
$6y^2z^2, \quad 2y^3z$					
$24x^4y^3, \quad 36x^3y^4$					
$12x^3y^2z^3, \quad 6x^2y^3z$					
$16x^2y^2z^2, \quad 12yz^3$					
$5x^3yz^3, \quad 7x^4y^2z^2, \quad 9x^2y^3z^3$					

B. Factoring Polynomials Whose Terms Have Common Factors

In Lesson D, you were shown the procedure for multiplying a polynomial by a monomial. For example, the product of $2x(3x^2 + 5x - 4)$ can be determined by applying the distributive property.

i.e. $$\begin{aligned} 2x(3x^2 + 5x - 4) &= (2x)(3x^2) + (2x)(5x) + (2x)(-4) \\ &= 6x^3 + 10x^2 - 8x \end{aligned}$$

Suppose you were interested in reversing this procedure and wished to write the polynomial $6x^3 + 10x^2 - 8x$ in factored form. You would proceed in the following manner.

1. Find the H.C.F. of all the terms in the polynomial. In this case, the H.C.F. of $6x^3$, $10x^2$, and $-8x$ is $2x$.

2. Write each term in the polynomial as the product of this H.C.F. and the factor which remains.
$$6x^3 + 10x^2 - 8x = \underset{\text{H.C.F.}}{\underset{\uparrow}{(2x)}}\underset{\substack{\uparrow\\ \text{Remaining Factor}\\ \text{(Quotient of } 6x^3 \text{ and } 2x)}}{(3x^2)} + (2x)(5x) + (2x)(-4)$$

3. Apply the distributive property in reverse. Take the common factor $2x$ out of each term in the polynomial.
$$\begin{aligned} 6x^3 + 10x^2 - 8x &= (2x)(3x^2) + (2x)(5x) + (2x)(-4) \\ &= 2x(3x^2 + 5x - 4) \end{aligned}$$

4. Check your work. Multiply these factors and make sure you obtain the original polynomial.
$$\begin{aligned} 2x(3x^2 + 5x - 4) &= 2x(3x^2) + 2x(5x) + 2x(-4) \\ &= 6x^3 + 10x^2 - 8x \end{aligned}$$

Any polynomial whose terms contain a common factor can be factored in this manner.

EXAMPLE: Factor the polynomial $9m^3n^2 - 27m^2n^3 - 3m^2n^2$.

Solution

Note that the H.C.F. of $9m^3n^2$, $-27m^2n^3$, and $-3m^2n^2$ is $3m^2n^2$.
Write each term in the polynomial as the product of $3m^2n^2$ and the factor which remains. (The remaining factor can be found by division.)

$$9m^3n^2 - 27m^2n^3 - 3m^2n^2 = \overset{\text{H.C.F.}}{\overset{\downarrow}{(3m^2n^2)}}\underset{\substack{\uparrow\\ \text{REMAINING FACTORS}}}{(3m)} + \overset{\downarrow}{(3m^2n^2)}\underset{\uparrow}{(-9n)} + \overset{\downarrow}{(3m^2n^2)}\underset{\uparrow}{(-1)}$$

Apply the distributive property in reverse and take the common factor $3m^2n^2$ out of each term in the polynomial.

$$= 3m^2n^2(3m - 9n - 1)$$

Check

$$\begin{aligned} 3m^2n^2(3m - 9n - 1) &= 3m^2n^2(3m) + 3m^2n^2(-9n) + 3m^2n^2(-1) \\ &= 9m^3n^2 - 27m^2n^3 - 3m^2n^2 \end{aligned}$$

Now, factor each of the following polynomials by taking a common factor out of every term.

1. $20x + 15 = 5(4x) + 5(3) = 5(\_\_\_ + \_\_\_)$

2. $2x^2 + 4xy = 2x(x) + 2x(\_\_\_) = 2x(x + \_\_\_)$

3. $12abc^2 - 6a^2bc = 6abc\ (\_\_\_) + \_\_\_(-a) = 6abc(\_\_\_ - a)$

4. $3b + 6bc - 15b^2 = 3b(1) + 3b(\_\_\_) + 3b(\_\_\_) = \_\_\_(1 + \_\_\_ - \_\_\_)$

Self-correcting Exercise #2

Answers may be found on page 63 of this lesson.

1. Fill in the missing factor in each case to make the statement true.

   (a) $5x^2y = 5x(\qquad)$
   (b) $-24a^2b^3 = -8ab^2(\qquad)$
   (c) $20ab^2c^3 = 4bc(\qquad)$
   (d) $-14cd^3e^2 = -2cde(\qquad)$
   (e) $18mn^4p^3 = 6n^2p^2(\qquad)$
   (f) $-21r^5s^3t^2 = 7r^2s^2t^2(\qquad)$
   (g) $-3abc = 3abc(\qquad)$
   (h) $-39u^6v^4w^2 = 3u^4v^3(\qquad)$
   (i) $2xy^2z = y^2z(\qquad)$
   (j) $2mn = 2mn(\qquad)$

2. State the H.C.F. of the terms in each polynomial.

	H.C.F.
(a) $2a^2 + 12a$	________
(b) $9b^2 - 81b$	________
(c) $12c^2 - 6$	________
(d) $mn^2t - m^2nt^2$	________

H.C.F.

2. (e) $9y^5 - 6y^4 + 3y^3$ __________

(f) $64x^3y^2 + 16xy^4 + 32x^2y^2$ __________

(g) $3x^2 - 12x + 18$ __________

(h) $abc + bc + ab$ __________

(i) $15x^3 - 20x^2 + 10x$ __________

(j) $x^2y + 2xy^2 + y^3$ __________

(k) $9a^5 - 36a^3b^2$ __________

(l) $(a + b)x + (a + b)y$ __________

3. Fill in the blanks in the following chart.

Polynomial	H.C.F. of Terms	Factorization of Terms	Polynomial in Factored Form
$b^3 - b^2 + b$	b	$b(b^2) + b(-b) + b(1)$	$b(b^2 - b + 1)$
$12y^4 + 30y^3 - 42y$			
$10a^2b - 15ab^2$			
$x^2 + 7x$			
$3x^2 - 21x^3$			
$12x^4 - 8x^2 + 20x$			
$5r^2t - 10rt + 5rt^2$			

4. Write the following polynomials in factored form by taking out common factors.

(a) $9b^2 - 81b$

=

(b) $9y^5 - 6y^4 + 3y^3$

=

(c) $12a^2b^2 + 6ab^2 - 3ab$ (d) $3ab - 5b + 7b^2$

$=$ $=$

C. Factoring Binomials Which Are the Differences of Squares

On pages 46 and 47 of lesson D, you were shown the procedure for multiplying two binomials which have the same first terms, and second terms which are additive inverse of each other. Remember that such products can always be written as the difference of two squares. For example, the product $(6a - 5b^2)(6a + 5b^2)$ can be determined by squaring the first term of either binomial, squaring the second term of either binomial, and subtracting these two squares.

$$\text{i.e.}\quad (6a - 5b^2)(6a + 5b^2) = (6a)^2 - (5b^2)^2$$
$$= 36a^2 - 25b^4$$

Suppose you were interested in reversing this procedure and wished to write the binomial $36a^2 - 25b^4$ in factored form. You would proceed in the following manner.

1. Examine the two terms of the binomial $36a^2 - 25b^4$ to see if they are perfect squares. Note that $36a^2$ is the square of $6a$ and that $25b^4$ is the square of $5b^2$. Also note that these terms are subtracted from each other.

2. Rewrite the binomial $36a^2 - 25b^4$ so that it appears as the difference of two squares.

$$36a^2 - 25b^4 = (6a)^2 - (5b^2)^2$$

3. Write the factors of the difference of two squares. These factors are the sum and difference of the two numbers which are squared. In this case, the factors of $(6a)^2 - (5b^2)^2$ are two binomials which are the sum and difference of $6a$ and $5b^2$.

$$(6a)^2 - (5b^2)^2 = \underbrace{(6a + 5b^2)}_{\text{(sum of 6a and } 5b^2\text{)}}\underbrace{(6a - 5b^2)}_{\text{(difference of 6a and } 5b^2\text{)}}$$

4. Check your work. Multiply these factors and make sure you obtain the original binomial.

$$(6a + 5b^2)(6a - 5b^2) = (6a)^2 - (5b^2)^2 = 36a^2 - 25b^4$$

Any binomial which can be expressed as the difference of two squares can be factored in this manner. (We have no way of factoring a binomial which is the sum of squares. For example, $x^2 + y^2$, $x^4 + y^4$, $4m^2 + 1$, are all prime polynomials.)

EXAMPLE: Factor the binomial $49a^2b^4 - 9c^6$.

Solution

Note that $49a^2b^4$ is the square of $7ab^2$, $9c^6$ is the square of $3c^3$, and these two perfect squares are subtracted from each other. Therefore, this binomial can be factored as the difference of two squares. The factors are the sum and difference of the two numbers that are squared.

$$\begin{aligned} 49a^2b^4 - 9c^6 &= \underbrace{(7ab^2)^2 - (3c^3)^2}_{\text{DIFFERENCE OF SQUARES}} \\ &= \underline{\underline{(7ab^2 \underset{\substack{\uparrow \\ \text{SUM}}}{+} 3c^3)(7ab^2 \underset{\substack{\uparrow \\ \text{DIFFERENCE}}}{-} 3c^3)}} \end{aligned}$$

Check

$$\begin{aligned} (7ab^2 + 3c^3)(7ab^2 - 3c^3) &= (7ab^2)^2 - (3c^3)^2 \\ &= 7^2a^2(b^2)^2 - 3^2(c^3)^2 \\ &= 49a^2b^4 - 9c^6 \end{aligned}$$

Now, factor the following binomials as the difference of squares.

1. $y^2 - 16 = y^2 - \underline{\quad}^2 = (y - 4)(y + \underline{\quad})$
2. $9t^2 - 25r^2 = (3t)^2 - (5\underline{\quad})^2 = (3t - 5r)(3t + \underline{\quad})$
3. $36t^2 - 1 = (\underline{\quad})^2 - 1^2 = (\underline{\quad} - \underline{\quad})(\underline{\quad} + \underline{\quad})$
4. $m^6 - n^2 = (m^3)^2 - \underline{\quad}^2 = (m^3 - \underline{\quad})(m^3 + \underline{\quad})$
5. $64y^2 - 49x^2 = (\underline{\quad})^2 - (\underline{\quad})^2 = (\underline{\quad} - \underline{\quad})(\underline{\quad} + \underline{\quad})$
6. $4a^2b^2 - \frac{1}{9} = (\quad)^2 - (\quad)^2 = (\underline{\quad} - \underline{\quad})(\underline{\quad} + \underline{\quad})$

In some cases, you will not recognize a binomial as being the difference of two squares until you remove a common factor from each of the two terms. When factoring, always look for common factors first.

EXAMPLE: Factor the binomial $27ax^2 - 12ay^2$

Solution

As it is written, the binomial $27ax^2 - 12ay^2$ cannot be factored as the difference of two squares since neither of its terms are perfect squares. But, note that the factor 3a is common to both terms. Once the factor 3a is removed from both terms, the remaining factors are perfect squares.

$$\begin{aligned} 27ax^2 - 12ay^2 &= (3a)(9x^2) + (3a)(-4y^2) \\ &= 3a\underbrace{(9x^2 - 4y^2)}_{\text{DIFFERENCE OF SQUARES}} \\ &= 3a[(3x)^2 - (2y)^2] \\ &= \underline{\underline{3a(3x + 2y)(3x - 2y)}} \end{aligned}$$

Check

$$\begin{aligned} 3a(3x + 2y)(3x - 2y) &= 3a[(3x)^2 - (2y)^2] \\ &= 3a[(3)^2(x)^2 - (2)^2(y)^2] \\ &= 3a(9x^2 - 4y^2) \\ &= 3a(9x^2) + (3a)(-4y^2) \\ &= 27ax^2 - 12ay^2 \end{aligned}$$

Remove the common factor from each term in the binomial below, and then factor the binomial as the difference of two squares.

$$\begin{aligned} 12mx^3 - 75mxy^2 &= (3mx)(\underline{\quad}) + (\underline{\quad})(-25y^2) \\ &= 3mx(\underline{\quad} - 25y^2) \\ &= 3mx[(\underline{\quad})^2 - (\underline{\quad})^2] \\ &= 3mx(\underline{\quad} - \underline{\quad})(\underline{\quad} + \underline{\quad}) \end{aligned}$$

Self-correcting Exercise #3

Answers may be found on page 64 of this lesson.

1. Express each binomial as the difference of two squares.

EXAMPLE: $4m^2n^4 - 9 = (2mn^2)^2 - (3)^2$

(a) $81x^2 - 100y^2$

(b) $a^2b^2 - 1$

(c) $64a^2 - b^2$

(d) $1 - 16y^4$

(e) $9a^8 - 25b^4$

(f) $x^4y^6 - 225z^{10}$

2. Write the following binomials in factored form. Look for common factors first.

(a) $20a^3b - 5ab^3$
$= (5ab)(4a^2) + (5ab)(-b^2)$
$= 5ab(4a^2 - b^2)$

$=$

(b) $mx^4 - my^4$

(c) $16s^2 - t^2$

(d) $16m^4n^2 - p^6$

(e) $ay^4 - a$

(f) $50x^4 - 32y^4$

(g) $(x^2 + 1)^2 - x^2$

(h) $a^2 - (b + c)^2$

D. Factoring Trinomials Which Are Perfect Squares

On pages 49 and 50 of Lesson D, you were shown the procedure for squaring a binomial. Remember that such products are trinomials with the following characteristics.

1. The first term is the square of the first term of the binomial.
2. The middle term is twice the product of the two terms of the binomial.
3. The last term is the square of the second term of the binomial.

For example, the binomial $(2a - 3b)^2$ can be expanded by squaring 2a, taking twice the product of 2a and -3b, squaring -3b, and adding these three results.

$$\begin{aligned} \text{i.e.}\quad (2a - 3b)^2 &= (2a)^2 + 2(2a)(-3b) + (-3b)^2 \\ &= 4a^2 - 12ab + 9b^2 \end{aligned}$$

NOTE THAT THESE SIGNS CORRESPOND.

Suppose you were interested in reversing this procedure and wished to write the perfect square trinomial $4a^2 - 12ab + 9b^2$ in factored form. You would proceed in the following manner.

1. Examine the three terms of the trinomial $4a^2 - 12ab + 9b^2$ to see if they satisfy the conditions of a perfect square. Note that $4a^2$ is the square of 2a, $9b^2$ is the square of -3b, and -12ab is twice the product of 2a and -3b.

2. Write the trinomial as the square of the binomial (2a - 3b).

$$\begin{aligned} 4a^2 - 12ab + 9b^2 &= (2a)^2 + 2(2a)(-3b) + (-3b)^2 \\ &= (2a - 3b)^2 \end{aligned}$$

3. Check your work. Square the binomial and make sure you obtain the original trinomial.

$$\begin{aligned} (2a - 3b)^2 &= (2a)^2 + 2(2a)(-3b) + (-3b)^2 \\ &= 4a^2 - 12ab + 9b^2 \end{aligned}$$

Any trinomial which is a perfect square can be factored in this manner.

EXAMPLE: Factor the trinomial $25x^2 + 80xy + 64y^2$.

Solution

This is a perfect square trinomial since $25x^2$ is the square of 5x, $64y^2$ is the square of 8y, and 80xy is twice the product of 5x and 8y.

$$\begin{aligned} 25x^2 + 80xy + 64y^2 &= (5x)^2 + 2(5x)(8y) + (8y)^2 \\ &= \underline{\underline{(5x + 8y)^2}} \end{aligned}$$

Check

$$\begin{aligned} (5x + 8y)^2 &= (5x)^2 + 2(5x)(8y) + (8y)^2 \\ &= 25x^2 + 80xy + 64y^2 \end{aligned}$$

Now, write each trinomial below as the square of a binomial.

1. $25a^2 - 10ab + b^2 = (5a)^2 + 2(5a)(-b) + (-b)^2 = (\_\_\_ - \_\_\_)^2$

2. $x^2 + 14x + 49 = \_\_\_^2 + 2(\_\_\_)(\_\_\_) + (\_\_\_)^2 = (\_\_\_ + \_\_\_)^2$

3. $9x^2 - 24x + 16 = (\_\_\_)^2 + \_\_\_(\_\_\_)(-4) + (\_\_\_)^2 = (\_\_\_ - \_\_\_)^2$

4. $x^2 - 6xy + 9y^2 = \_\_\_^2 + 2(\_\_\_)(\_\_\_) + (\_\_\_)^2 = (\_\_\_\_\_\_\_\_\_)^2$

5. $x^4 + 12x^2y^2 + 36y^4 = (x^2)^2 + \_\_\_\_\_\_\_\_\_ + (\_\_\_)^2 = (\_\_\_\_\_\_\_\_\_)^2$

In some cases, you will not recognize a perfect square trinomial until you remove a common factor from all three terms. Remember to always look for common factors first.

EXAMPLE: Factor the trinomial $50x^3 - 80x^2 + 32x$.

Solution

As it is written, this trinomial is not a perfect square. But, if the common factor $2x$ is removed from each term, it becomes a perfect square.

$$\begin{aligned} 50x^3 - 80x^2 + 32x &= 2x(25x^2) + 2x(-40x) + 2x(16) \\ &= 2x(\underbrace{25x^2 - 40x + 16}_{\text{PERFECT SQUARE TRINOMIAL}}) \\ &= 2x[(5x)^2 + 2(5x)(-4) + (-4)^2] \\ &= \underline{\underline{2x(5x - 4)^2}} \end{aligned}$$

Check

$$\begin{aligned} 2x(5x - 4)^2 &= 2x[(5x)^2 + 2(5x)(-4) + (-4)^2] \\ &= 2x(25x^2 - 40x + 16) \\ &= 2x(25x^2) + 2x(-40x) + 2x(16) \\ &= 50x^3 - 80x^2 + 32x \end{aligned}$$

Remove the common factor from each term in the trinomial below, and then write the trinomial as the square of a binomial.

$$\begin{aligned} 45a^3b + 30a^2b^2 + 5ab^3 &= (5ab)(\_\_\_) + (5ab)(\_\_\_) + (\_\_\_)(\_\_\_) \\ &= \_\_\_(\_\_\_ + \_\_\_ + \_\_\_) \\ &= \_\_\_[(\_\_\_)^2 + 2(\_\_\_)(\_\_\_) + (\_\_\_)^2] \\ &= \_\_\_(\_\_\_\_\_\_\_\_\_)^2 \end{aligned}$$

Self-correcting Exercise #4

Answers may be found on page 64 of this lesson.

1. Decide whether or not each of the following trinomials is a perfect square.

Yes or No?

(a) $x^2 + 8x + 16$ ________

(b) $p^2 - 2p + 1$ ________

(c) $16x^2 - 12x + 9$ ________

(d) $a^2 + 2a + b^2$ ________

(e) $r^2 - 2rs + s^2$ ________

(f) $9t^2 + 30t + 25$ ________

(g) $2x^2 + 12x + 3$ ________

(h) $4x^4 + 12x^2y^2 + 9y^4$ ________

(i) $z^2 + 2z - 1$ ________

2. Make the trinomials perfect squares by filling in the blanks.

(a) $a^2 + \_\_\_\_ + 25 = (\qquad)^2$

(b) $16x^2 + \_\_\_\_ + 9 = (\qquad)^2$

(c) $9y^2 - \_\_\_\_ + 1 = (\qquad)^2$

(d) $4x^2 + 12x + \_\_\_\_ = (\qquad)^2$

(e) $\_\_\_\_ - 16x + 64 = (\qquad)^2$

(f) $9a^2 - 12ab + \_\_\_\_ = (\qquad)^2$

(g) $\_\_\_\_ - 30x + 25 = (\qquad)^2$

(h) $16x^2y^2 - \_\_\_\_ + 1 = (\qquad)^2$

3. Write the following trinomials in factored form. Look for common factors first.

(a) $2n^3 + 8n^2 + 8n$
$= 2n(n^2 + 4n + 4)$

$=$

(b) $27x^2 - 18x + 3$

(c) $b^2 + 14b + 49$

(d) $16x^2 - 8x + 1$

(e) $144n^2 + 120nx + 25x^2$ (f) $243m^2n^2 + 54mn + 3$

(g) $16x^2 - 24xy + 9y^2$ (h) $x^4 + 2x^2y^2 + y^4$

E. Factoring Trinomials of the Form $x^2 + bx + c$

Whenever we multiply two binomials of the form $(x + r)$ and $(x + s)$ where r and s are integers, the result is a quadratic trinomial in x.

$$\begin{aligned}(x + r)(x + s) &= x^2 + rx + sx + rs \\ &= x^2 + (r + s)x + rs\end{aligned}$$

Note that in the above product the x^2-term has a numerical coefficient of 1, the x-term has a numerical coefficient of $(r + s)$, and the constant term is rs.

Study the following products.

1. $(x + 3)(x + 2) = x^2 + 5x + 6$
   The two integers in the binomial factors are 3 and 2. The sum of these integers is 5, the coefficient of x in the product. The product of these integers is 6, the constant term in the product.

2. $(x - 3)(x + 2) = x^2 - x - 6$
   The two integers in the binomial factors are -3 and 2. The sum of these integers is -1, the coefficient of x in the product. The product of these integers is -6, the constant term in the product.

3. $(x + 3)(x - 2) = x^2 + x - 6$
   The two integers in the binomial factors are 3 and -2. The sum of these integers is 1, the coefficient of x in the product. The product of these integers is -6, the constant term in the product.

4. $(x - 3)(x - 2) = x^2 - 5x + 6$
   The two integers in the binomial factors are -3 and -2. The sum of these integers is -5, the coefficient of x in the product. The product of these integers is 6, the constant term in the product.

Suppose you were interested in reversing this procedure and wished to write the trinomials $x^2 + 5x + 6$, $x^2 - x - 6$, $x^2 + x - 6$, and $x^2 - 5x + 6$ in factored form. You would proceed as follows.

1. In the trinomial $x^2 + 5x + 6$, note that the coefficient of x is 5 and the constant term is 6. You must find two integers with a sum of 5 and a product of 6. Since the product is positive, both integers must have the same sign. Since the sum is positive, both integers must be positive. The two integers must be 2 and 3 since $2 + 3 = 5$ and $2 \times 3 = 6$.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

(Two integers with a sum of 5 and product of 6.)

2. In the trinomial $x^2 - x - 6$, note that the coefficient of x is -1 and the constant term is -6. You must find two integers with a sum of -1 and a product of -6. Since the product is negative, the integers must have opposite signs. Since the sum is negative, the larger integer must be negative. The two integers must be 2 and -3 since $2 + (-3) = -1$ and $2 \times -3 = -6$.

$$x^2 - x - 6 = (x + 2)(x - 3)$$

(Two integers with a sum of -1 and a product of -6.)

3. In the trinomial $x^2 + x - 6$, note that the coefficient of x is 1 and the constant term is -6. You must find two integers with a sum of 1 and a product of -6. Since the product is negative, the integers must have opposite signs. Since the sum is positive, the larger integer must be positive. The two integers must be -2 and 3 since $-2 + 3 = 1$ and $-2 \times 3 = -6$.

$$x^2 + x - 6 = (x - 2)(x + 3)$$

(Two integers with a sum of 1 and a product of -6.)

4. In the trinomial $x^2 - 5x + 6$, note that the coefficient of x is -5 and the constant term is 6. You must find two integers with a sum of -5 and a product of 6. Since the product is positive, both integers must have the same sign. Since the sum is negative, both integers must be negative. The two integers must be -2 and -3 since $(-2) + (-3) = -5$ and $-2 \times -3 = 6$.

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

(Two integers with a sum of -5 and a product of 6.)

Thus, when factoring any trinomial of the form $x^2 + bx + c$, you must look for two integers that give a sum of b and product of c. The following rules can be used to help you determine the signs of these integers.

1. If both b and c are positive, both integers involved in the factors must be positive.

   e.g. $x^2 + 3x + 2 = (x + 2)(x + 1)$

   b + c ARE POSITIVE.

   INTEGERS IN FACTORS ARE POSITIVE.

   The integers +2 and +1 give a sum of +3 and product of +2.

2. If b is negative and c is positive both integers involved in the factors must be negative.

   e.g. $x^2 - 3x + 2 = (x - 2)(x - 1)$

   b IS NEGATIVE.
   c IS POSITIVE.

   INTEGERS IN FACTORS ARE NEGATIVE.

   The integers -2 and -1 give a sum of -3 and product of +2.

3. If b is positive and c is negative, the integers involved in the factors must have unlike signs. The integer with the larger absolute value will be positive.

   e.g. $x^2 + 3x - 4 = (x + 4)(x - 1)$

   b IS POSITIVE.
   c IS NEGATIVE.

   INTEGERS WITH LARGER ABSOLUTE VALUE IS POSITIVE. (OTHER INTEGER IS NEGATIVE.)

   The integers +4 and -1 have a sum of +3 and product of -4.

4. If both b and c are negative, the integers involved in the factors must have unlike signs. The integer with the larger absolute value will be negative.

   e.g. $x^2 - 2x - 15 = (x + 3)(x - 5)$

   b AND c ARE NEGATIVE.

   INTEGER WITH LARGER ABSOLUTE VALUE IS NEGATIVE. (OTHER INTEGER IS POSITIVE)

   The integers +3 and -5 have a sum of -2 and product of -15.

Finish factoring the trinomials below by inserting the correct signs.

1. $y^2 - y - 12 = (y - 4)(y + 3)$
2. $a^2 - 4a + 3 = (a \quad 3)(a \quad 1)$
3. $x^2 + 3xy - 10y^2 = (x \quad 5y)(x \quad 2y)$
4. $x^2 - 6x - 7 = (x \quad 1)(x \quad 7)$
5. $x^2 - 4x - 32 = (x \quad 8)(x \quad 4)$
6. $m^2 + 9m + 14 = (m \quad 7)(m \quad 2)$
7. $x^2 - 11x + 30 = (x \quad 6)(x \quad 5)$
8. $a^2 - 4ab - 12b^2 = (a \quad 6b)(a \quad 2b)$

Finish factoring the trinomials below by inserting the correct numbers.

1. $x^2 - 4x + 3 = (x - 3)(x - 1)$ since -3 and -1 give a sum of -4 and a product of 3.

2. $t^2 + 2t - 15 = (t + \_\_)(t - \_\_)$ since 5 and -3 give a sum of 2 and a product of -15.

3. $r^2 - 3r - 28 = (r + \_\_)(r - \_\_)$ since __ and __ give a sum of -3 and a product of -28.

4. $y^2 - 9y + 20 = (y - \_\_)(y - \_\_)$ since __ and __ give a sum of -9 and a product of 20.

5. $x^2 - 5x - 14 = (x + \_\_)(x - \_\_)$ since __ and __ give a sum of -5 and a product of -14.

6. $x^2 - 9x + 18 = (x - \_\_)(x - \_\_)$ since __ and __ give a sum of -9 and a product of 18.

7. $t^2 - 9t - 22 = (t - \_\_)(t + \_\_)$ since __ and __ give a sum of -9 and a product of -22.

8. $x^2 + 2x - 35 = (x - \_\_)(x + \_\_)$ since __ and __ give a sum of 2 and a product of -35.

In some cases you will have to remove a common factor from every term before a trinomial is of the form $x^2 + bx + c$.

EXAMPLE: Factor the trinomial $2x^2 - 18x + 40$.

Solution

First, take the common factor 2 out of each term.

$$\begin{aligned} 2x^2 - 18x + 40 &= 2(x^2) + 2(-9x) + 2(20) \\ &= 2(x^2 - 9x + 20) \end{aligned}$$

Then, factor the trinomial $x^2 - 9x + 20$. (You must find two integers whose sum is -9 and product is 20.)

$$= \underline{\underline{2(x - 4)(x - 5)}}$$

Remove the common factor from every term in the trinomial below and then factor the trinomial.

$$\begin{aligned} 3x^2y + 9xy - 54y &= (3y)(\_\_\_) + (3y)(\_\_\_) + (3y)(\_\_\_) \\ &= 3y\ (\_\_\_ + \_\_\_ - \_\_\_) \\ &= 3y\ (\_\_\_\_\_\_)(\_\_\_\_\_\_) \end{aligned}$$

Self-correcting Exercise #5

Answers may be found on page 65 of this lesson.

1. Find pairs of integers with the following sums and products.

	SUM	PRODUCT	INTEGERS
(a)	10	24	
(b)	-10	24	
(c)	11	18	
(d)	15	50	
(e)	15	44	
(f)	3	-4	
(g)	-4	3	
(h)	-4	-96	
(i)	-19	60	

2. Factor the following trinomials. Look for common factors first.

(a) $x^2 - 4x - 12 =$

(b) $x^2 + 12x + 35 =$

(c) $x^2 - 2x - 63 =$

(d) $x^2 + 2x - 8 =$

(e) $x^2 - 13x - 30 =$

(f) $x^2 - 3x - 40 =$

(g) $x^2 + 8x + 15 =$

(h) $x^2 - x - 56 =$

MAKE COEFFICIENT OF x^2 ONE INSTEAD OF MINUS ONE.

(i) $-x^2 - 4x + 21 = -(x^2 + 4x - 21) = -(\qquad)(\qquad)$

(j) $-x^2 + 6x + 27 =$

(k) $3x^2 + 9x - 30 =$

(l) $-2x^2 + 2x + 4 = -2(\qquad) = -2(\qquad)(\qquad)$

F. Factoring Trinomials of the Form $ax^2 + bx + c$

Whenever we multiply two binomials of the form $(mx + r)$ and $(nx + s)$ where m, n, r, and s are integers, the result is a quadratic trinomial in x.

$$\begin{aligned}(mx + r)(nx + s) &= mnx^2 + rnx + msx + rs \\ &= mnx^2 + (rn + ms)x + rs\end{aligned}$$

Note that in the above product, the x^2-term has a numerical coefficient of mn, the x-term has a numerical coefficient of $(rn + ms)$, and the constant term is rs. The middle term of the product, $(rn + ms)x$, is often referred to as the CROSS-PRODUCT SUM (CPS).

Study the following products.

1. $(3x + 1)(2x + 5) = 6x^2 + 17x + 5$
   The first term in the product is $6x^2$ which is the product of 3x and 2x, the first terms in the binomial factors. The last term in the product is 5 which is the product of 1 and 5, the last terms in the binomial factors. The middle term in the product is the sum of 2x and 15x, the inner and outer products formed by these binomial factors. (i.e. the CPS is 17x.)

2. $(2x + 7)(x - 2) = 2x^2 + 3x - 14$
   The first term in the product is $2x^2$ which is the product of 2x and x, the first terms in the binomial factors. The last term in the product is -14 which is the product of 7 and -2, the last terms in the binomial factors. The middle term in the product is the sum of 7x and -4x, the inner and outer products formed by these binomial factors. (i.e. the CPS is 3x.)

3. $(12x - 5)(x - 3) = 12x^2 - 41x + 15$
   The first term in the product is $12x^2$ which is the product of 12x and x, the first terms in the binomial factors. The last term in the product is 15 which is the product of -5 and -3, the last terms in the binomial factors. The middle term in the product is the sum of -5x and -36x, the inner and outer products formed by these binomial factors. (i.e. the CPS is -41x.)

Suppose you were interested in reversing this procedure and wished to write the trinomials $6x^2 + 17x + 5$, $2x^2 + 3x - 14$, and $12x^2 - 41x + 15$ in factored form. You would proceed as follows.

1. In the trinomial $\underline{6x^2 + 17x + 5}$, note that the first term is $6x^2$. Thus, the first terms of the binomial factors must be $6x$ and x or $3x$ and $2x$.

$$(6x \qquad\qquad)(x \qquad\qquad)$$

$$(3x \qquad\qquad)(2x \qquad\qquad)$$

The constant term in the trinomial is 5. Thus, the second terms of the binomial factors must be two integers with like signs giving a product of 5. In fact, since the middle term of the trinomial is positive, we know that these two integers must both be positive. Thus, the second terms of the binomial factors must be 5 and 1.

$$(\qquad + \qquad 5)(\qquad + \qquad 1)$$

Below is a list of the possible combinations of these first and last terms. We must find the CPS of each of these pairs of factors in order to discover the pair that has a CPS of $17x$. (Recall that the trinomial you are trying to factor has a middle term of $17x$.)

	CPS	
$(6x + 5)(x + 1)$	$5x + 6x = 11x$	
$(6x + 1)(x + 5)$	$x + 30x = 31x$	
$(3x + 5)(2x + 1)$	$10x + 3x = 13x$	
$(3x + 1)(2x + 5)$	$2x + 15x = 17x$	*

Since the product $(3x +1)(2x + 5)$ has a CPS of $17x$, these must be the factors of the trinomial $6x^2 + 17x + 5$.

i.e. $6x^2 + 17x + 5 = \underline{\underline{(3x + 1)(2x + 5)}}$

2. In the trinomial $\underline{2x^2 + 3x - 14}$, note that the first term is $2x^2$. Thus, the first terms of the binomial factors must be $2x$ and x.

$$(2x \qquad\qquad)(x \qquad\qquad)$$

The constant term in the trinomial is -14. Thus, the second terms of the binomial factors must be two integers with unlike signs giving a product of -14. Thus, these terms must be 1 and -14, -1 and 14, -2 and 7, or 2 and -7.

$$\begin{array}{l} (\qquad + \qquad 1)(\qquad - \qquad 14) \\ (\qquad - \qquad 1)(\qquad + \qquad 14) \\ (\qquad - \qquad 2)(\qquad + \qquad \;\;7) \\ (\qquad + \qquad 2)(\qquad - \qquad \;\;7) \end{array}$$

Below is a list of the possible combinations of these first and last terms. State the CPS of each pair.

$(2x + 1)(x - 14)$ CPS is $-27x$.
$(2x - 14)(x + 1)$ CPS is _____.
$(2x - 1)(x + 14)$ CPS is _____.
$(2x + 14)(x - 1)$ CPS is _____.
$(2x - 2)(x + 7)$ CPS is _____.
$(2x + 7)(x - 2)$ CPS is _____.
$(2x + 2)(x - 7)$ CPS is _____.
$(2x - 7)(x + 2)$ CPS is _____.

Since the product $(2x + 7)(x - 2)$ has a CPS of $3x$, these must be the factors of the trinomial $2x^2 + 3x - 14$.

i.e. $2x^2 + 3x - 14 = \underline{\underline{(2x + 7)(x - 2)}}$

3. In the trinomial $\underline{12x^2 - 41x + 15}$, note that the first term is $12x^2$. Thus, the first terms of the binomial factors must be $12x$ and x, $4x$ and $3x$, or $6x$ and $2x$.

$(12x \quad)(x \quad)$
$(4x \quad)(3x \quad)$
$(6x \quad)(2x \quad)$

The constant term in the trinomial is 15. Thus, the second terms of the binomial factors must be two integers with like signs giving a product of 15. In fact, since the middle term of the trinomial is negative, we know that these two integers must both be negative. Thus, the second terms of the binomial factors must be -15 and -1 or -3 and -5.

Below is a list of the possible combinations of the first and last terms. State the CPS of each pair.

$(12x - 15)(x - 1)$ CPS is $-27x$.
$(12x - 1)(x - 15)$ CPS is _____.
$(12x - 3)(x - 5)$ CPS is _____.
$(12x - 5)(x - 3)$ CPS is _____.
$(4x - 15)(3x - 1)$ CPS is _____.
$(4x - 1)(3x - 15)$ CPS is _____.
$(4x - 3)(3x - 5)$ CPS is _____.
$(4x - 5)(3x - 3)$ CPS is _____.
$(6x - 15)(2x - 1)$ CPS is _____.
$(6x - 1)(2x - 15)$ CPS is _____.
$(6x - 3)(2x - 5)$ CPS is _____.
$(6x - 5)(2x - 3)$ CPS is _____.

Since the product $(12x - 5)(x - 3)$ has a CPS of $-41x$, these must be the factors of the trinomial $12x^2 - 41x + 15$.

i.e. $12x^2 - 41x + 15 = \underline{\underline{(12x - 5)(x - 3)}}$

Fill in the blanks below in order to factor the given trinomials.

1. $2x^2 + 7x + 3$

(a) Since the first term of the trinomial is _____, the first terms of the binomial factors must be _____ and _____.

(_____)(_____)

(b) Since the last term of the trinomial is _____, the second terms of the binomial factors must be two integers with like signs giving a product of _____. Since the middle term of the trinomial is positive, the two integers must be __________. Thus, the second terms of the binomial factors must be _____ and _____.

(_____)(_____)

(c) The pair of binomials that gives a CPS of 7x is:

(_____ + _____)(_____ + _____)

2. $4x^2 - 8x - 21$

(a) Since the first term of the trinomial is _____, the first terms of the binomial factors must be _____ and _____ or _____ and _____.

(_____)(_____)
(_____)(_____)

(b) Since the last term of the trinomial is _____, the second terms of the binomial factors must be two integers with __________ signs that give a product of _____. Thus, the second terms must be 1 and -21, -1 and _____, 3 and _____, or -3 and _____.

(+ 1)(- 21)
(_____)(_____)
(_____)(_____)
(- 3)(+ 7)

Refer to
#2, page
20

(c) Test possible combinations. Stop when you obtain a CPS of -8x.

Possible Factors	CPS
$(4x+1)(x-21)$	$-83x$

Thus, $4x^2 - 8x - 21 = ($ $)($ $)$.

3. $3x^2 + x - 30$

(a) Since the first term of the trinomial is _____, the first terms of the binomial factors must be _____ and _____.

(___)(___)

(b) Since the last term of the trinomial is _____, the second terms of the binomial factors must be two integers with __________ signs that give a product of _____. Thus, the second terms must be 1 and -30, -1 and _____, 3 and _____, -3 and _____, 5 and _____, or -5 and _____, -2 and 15, or 2 and -15.

(+ 1)(- 30) (___)(___)

(___)(___) (___)(___)

(___)(___) (___)(___)

(___)(___) (___)(___)

(c) Test possible combinations. Stop when you obtain a CPS of x.

Possible Factors	CPS
$(3x+1)(x-30)$	$-89x$

Thus, $3x^2 + x - 30 = ($ $)($ $)$.

The procedure for factoring trinomials of the form $ax^2 + bx + c$ involves a fair amount of trial and error work. All trials should be made on a separate sheet of paper and only the correct factors written in your book. As you gain more practice in factoring quadratic trinomials, you will find that you will often hit upon the right factors before having to try out all the possible combinations.

Finish factoring the trinomials below by inserting the correct signs.

1. $12x^2 + 7x - 10 = (3x \quad 2)(4x \quad 5)$
2. $24x^2 - 38x + 15 = (4x \quad 3)(6x \quad 5)$
3. $2y^2 + 13y + 15 = (2y \quad 3)(y \quad 5)$
4. $8x^2 + 10x - 3 = (4x \quad 1)(2x \quad 3)$
5. $10x^2 - 29x + 10 = (5x \quad 2)(2x \quad 5)$

Finish factoring the trinomials below by inserting the correct second terms.

1. $5x^2 - 14x + 8 = (5x \qquad)(x \qquad)$
2. $3x^2 + 7x - 6 = (3x \qquad)(x \qquad)$
3. $15x^2 - 2x - 24 = (5x \qquad)(3x \qquad)$
4. $12x^2 - 11x - 15 = (4x \qquad)(3x \qquad)$
5. $35x^2 + 13x - 12 = (7x \qquad)(5x \qquad)$

Always look for common factors first before trying to factor a quadratic trinomial.

EXAMPLE: Factor the trinomial $-24x^2 + 18x + 15$.
Solution. Take the common factor -3 out of every term. (Note that it is better to remove the factor -3 instead of 3 since your first term will then be positive.)

$$\begin{aligned} -24x^2 + 18x + 15 &= (-3)(8x^2) + (-3)(-6x) + (-3)(-5) \\ &= -3(8x^2 - 6x - 5) \\ &= \underline{\underline{-3(4x - 5)(2x + 1)}} \end{aligned}$$

Remove the common factor from each term in the trinomial below and then factor the trinomial.

$$4x^2y + 10xy - 24y = (\_y)(\_\_\_\_\_) + (\_y)(\_\_\_\_\_) + (\_y)(\_\_\_\_\_)$$

$$= \_y\,(\_\_\_\_\_ + \_\_\_\_\_ - \_\_\_\_\_)$$

$$= \_y\,(\qquad\qquad)(\qquad\qquad)$$

Self-correcting Exercise #6

Answers may be found on page 65 of this lesson.

1. Supply the missing terms or signs in the following expressions.

   (a) $6x^2 + 17x + 5 = (2x \quad 5)(3x + 1)$

   (b) $2y^2 + 13y + 15 = (2y + 3)(y + \quad)$

   (c) $12a^2 + 28a + 15 = (\quad + 5)(2a + 3)$

   (d) $x^2 + 3xy - 10y^2 = (x - \quad)(x + \quad)$

   (e) $12x^2 - 29x + 15 = (3x \quad 5)(4x \qquad)$

   (f) $2a^2 + 7ab + 3b^2 = (2a \qquad)(a \qquad)$

2. Give the second factor in each of the following.

   (a) $2x^2 + 11x + 12 = (2x + 3)(\qquad)$

   (b) $6x^2 + 19x + 15 = (3x + 5)(\qquad)$

   (c) $8x^2 + 10x - 3 = (4x - 1)(\qquad)$

   (d) $10x^2 - 29x + 10 = (2x - 5)(\qquad)$

   (e) $15x^2 - 29x - 14 = (5x + 2)(\qquad)$

   (f) $8x^2 + 14x - 15 = (4x - 3)(\qquad)$

3. Factor the following trinomials. Look for common factors first.

   (a) $2x^2 + x - 6 =$

   (b) $12x^2 - 8x - 15 =$

   (c) $12y^2 + 2y - 14 =$

   (d) $4x^2 + 12x + 9 =$

   (e) $4x^2 - 4x + 1 =$

   (f) $9x^2 - 36x + 27 =$

   (g) $-6x^2 + 5x + 4 =$

   (h) $6x^2 + 7xy - 5y^2 =$

   (i) $4x^4 - 4x^2 - 35 =$

   (j) $8y^2 + 8y - 30 =$

G. Summary - Factoring

There is a procedure to follow in factoring.

1. First, look for factors common to every term. Then try to factor remaining binomial or trinomial factors if possible.

2. If a factor is a binomial, decide if it is the difference of squares. You can factor such binomials.

3. If a factor is a trinomial, decide if it is a perfect square. You can factor perfect squares.

4. If a factor is a trinomial which is not a perfect square, assume that it is the product of binomials, and search for them. Of course, you cannot factor a prime trinomial. But never decide that an expression is prime until you have tried all the ways you know of factoring.

5. After you factor a polynomial, write all the factors, including any monomial factor. The monomial factor may be a product, but all other factors should be prime; the factoring should be complete.

6. Always multiply the factors to see whether the product is equal to the original expression.

EXERCISE - Factoring

1. Fill in the blanks.

(a) The highest common factor of 36 and 40 is ______.

(b) When we factor a polynomial, we change an indicated ________________ to an indicated ________________.

(c) A ____________ polynomial is a polynomial which cannot be factored further.

(d) The highest common factor of all the terms in the polynomial $25x^2y - 10x^3y^2 + 15x^2y^2$ is ______________________.

(e) A trinomial of the form $ax^2 + bx + c$, where $a \neq 0$ is a ____________________ trinomial in x.

(f) When factoring the polynomial $x^2 + 6x + 5$, we must look for two integers whose sum is ______ and whose product is ______. The two integers are ______ and ______.

(g) $16x^2 +$ ▨ $+ 25$ would be a perfect square if the middle term were __________.

(h) The factors of $6x^2 - 7x - 5$ must have a CPS of _____.

(i) $x^4 - y^2$ can be factored as the _________ of two _________.

2. Decide if each of the following statements is true or false.

True or False?

(a) 3 × 5 is the prime factorization of 15. ________________

(b) $5x + 4$ is factorable over the integers. ________________

(c) $4(a^2 + ab)$ is a complete factorization of $4a^2 + 4ab$. ________________

(d) $(x - y)$ is a factor of $x^2 - 2xy + y^2$. ________________

True or False?

(e) $6\left(\frac{1}{2}x - 1\right)$ is a factorization over the integers of $3x - 6$. ________

(f) $x^2 + 7x + 12$ is factorable over the integers. (i.e. $x^2 + 7x + 12$ can be written as the product of factors with integral coefficients and constants.) ________

(g) $y^2 - 5y - 8$ is factorable over the integers. ________

(h) $3x^3 + 4x^2 - 5x$ is a quadratic in x. ________

(i) $(x + 3)(2x + 4)$ is a complete factorization of $2x^2 + 10x + 12$. ________

(j) $4t^2 + 12t + 36$ is a perfect square trinomial. ________

(k) $(4r - 3)$ is a factor of $4r^2 - 9$. ________

(l) $(2y + 5)(y - 2)$ is a factorization of $2y^2 + 3y - 10$. ________

(m) $(3x - 1)$ is a factor of $12x^2 - 7x + 1$. ________

(n) 3c is a common factor of the terms of $6bc + 9c^2 - 3$. ________

(o) $x^2 + y^2$ is a prime polynomial. ________

3. Factor each polynomial by taking the highest common factor out of every term.

(a) $45a^4b^5 - 18a^6b^3 = (9a^4b^3)(5b^2) + (9a^4b^3)(-2a^2) = 9a^4b^3(\underline{\quad\quad} - \underline{\quad\quad})$

(b) $3x - 6 =$ ________

(c) $x^2 - xy =$ ________

(d) $a^2 - ab - a =$ ________

(e) $6r^3 + 12r^4 + 24r^5 =$ ________

(f) $14y^3 - 21y^2 + 49y^6 =$ ________

(g) $80a^2 - 16ab =$ ________

4. Factor as the difference of squares. Look for common factors first.

(a) $p^2 - q^2$

$= (p-q)(p+q)$

(b) $2a^3 - 50ab^2$

(c) $25x^2 - 36y^2$

(d) $10x^3 - 90x$

(e) $a^6 - 9b^4$

(f) $(x + y)^2 - z^2$

(g) $27ax^5 - 3a^3x$

(h) $m^2 - (n + 2)^2$

5. Factor the following perfect square trinomials.

(a) $4n^2 + 28n + 49$

(b) $9a^2b^2 - 30ab + 25$

(c) $25x^2 - 90xy + 81y^2$

(d) $9x^4 + 6x^2 + 1$

6. Factor the following trinomials. Look for common factors first.

(a) $4x^2 + 10x + 4$
$= 2(\qquad\qquad)$
$= 2(\qquad)(\qquad)$

(b) $x^2 - 6x - 27$

(c) $x^2 - 15x + 44$

(d) $3x^2 + 9x - 84$

(e) $12y^2 - 5y - 3$

(f) $3a^2 - 5a + 2$

(g) $2x^2 - x - 3$

(h) $4a^2 + 13ab + 3b^2$
$= (\quad + b)(\quad + 3b)$

(i) $-x^2 - 3x + 10$
$= -(x^2 + 3x - 10)$
$=$

(j) $-6x^2 + x + 2$

(k) $30x^2 + 10x - 100$

(l) $m^3 + 5m^2 - 36m$

(m) $3a^2 - 23ab - 36b^2$

(n) $10x^4 - x^2 - 2$

7. Factor each polynomial completely. Look for common factors first.

(a) $x^2 - x - 56$

(b) $x^3 - x$

(c) $a^4 - 1$

(d) $2x^2 - 12x + 18$

(e) $4x^2 - 3x - 1$

(f) $27x^3 - 3x$

(g) $100b^4 - 20b^2 + 1$

(h) $20a^2 + 10ab - 30b^2$

(i) $xy - x^3y^3$

(j) $ax^2 + 4ax - 5a$

(k) $16y^4 - 12y^3 - 18y^2$

(l) $10x^2y^2 + 7xy^2 - 12y^2$

(m) $16x^2 - 15x - 1$

(n) $5 - 80a^2$

(o) $a^3b^3 - 2a^2b^2 - 3ab$

(p) $-5y^2 - 6y + 11$

(q) $a^5 - 81a$

(r) $(x + y)^2 - 1$

(s) $6u^2v^2 - 9uv^3 + 3uv^2$

(t) $4x^2 - 12x + 9$

(u) $64y^2 - 49x^2$

(v) $12y^2 + 5y - 3$

Topic Two: Rational Expressions

A. Definition of a Rational Expression

In Lesson B, we defined a RATIONAL NUMBER to be any number that can be written as the quotient of two integers. The following are all examples of rational numbers.

$$\frac{-2}{5}, \quad \frac{6}{7}, \quad \frac{0}{-9}, \quad \frac{-4}{-1}$$

Remember that the denominator of a rational number cannot be zero since division by zero is impossible.
For example,

$\frac{2}{0}$, $\frac{-4}{0}$, $\frac{0}{0}$ are not rational numbers.

In a corresponding manner, we may define a RATIONAL EXPRESSION to be any expression that may be written as the quotient of two polynomials. The following are all examples of rational expressions.

$$\frac{2x}{y}, \quad \frac{5}{x - 1}, \quad \frac{x^2 - 3x + 2}{x^2 - 1}, \quad \frac{a + b}{1}$$

Since division by zero is impossible, a rational expression is not defined for any replacements of the variables which make the denominator equal to zero. For example,

$\frac{2x}{y}$ is not defined when $y = 0$. The denominator, y, will equal zero when $y = 0$.

$\frac{5}{x - 1}$ is not defined when $x = 1$. The denominator, $x - 1$, will equal zero when $x = 1$.

$\frac{x^2 - 3x + 2}{x^2 - 1}$ is not defined when $x = 1$ or $x = -1$. The denominator, $x^2 - 1$, which equals $(x - 1)(x + 1)$ will equal zero when $x = 1$ or $x = -1$.

$\frac{a + b}{1}$ is defined for all values of "a" and "b". The denominator has a constant value of 1.

It is important to note that every polynomial is a rational expression with a denominator of 1 (just as every integer is a rational number with a denominator of 1). For example, the polynomials 3, x, $x^2 + 1$, $x + y - z$, can be written in rational form as follows:

$$\frac{3}{1}, \quad \frac{x}{1}, \quad \frac{x^2 + 1}{1}, \quad \frac{x + y - z}{1}$$

Self-correcting Exercise #7

Answers may be found on page 66 of this lesson.

1. Decide whether or not each of the following is a rational expression. Put "yes" or "no" in each blank.

(a) $\frac{a + b}{c + d}$ = ______

(b) $\frac{4x^{\frac{2}{3}} + y^2}{z}$ ______

(c) $x^2 + 2x + 3$ ______

(d) $\frac{1}{x + y}$ ______

(e) $x + 5 = y$ ______

(f) $\frac{6y}{\sqrt{x}}$ ______

2. State the values of x for which the following rational expressions are not defined. (Factor any denominator that contains more than one term.)

(a) $\frac{1}{x^2}$ (b) $\frac{x + 4}{(x - 2)(x + 1)}$ (c) $\frac{x^2 + 2}{x(x - 3)}$

__________ __________ __________

(d) $\frac{5}{x^2 + 3x}$ (e) $\frac{x^2 + 2x + 1}{x^2 - 3x - 10}$ (f) $\frac{x^2 + 3}{x^2 - 16}$

__________ __________ __________

B. Reducing Rational Expressions to Lowest Terms

You will recall that a rational number is expressed in lowest terms when its numerator and denominator have no common factors other than 1 and -1. The Principle of Equivalent Fractions (P.E.F.) allows us to write a rational number in lowest terms by dividing its numerator and denominator by their highest common factor.

EXAMPLE: Write the rational number $\frac{72}{108}$ in lowest terms.

Solution

The H.C.F. of 72 and 108 is 36. (i.e. the largest number that will divide into both the numerator and denominator of the rational number $\frac{72}{108}$ is 36.) Divide both terms of the fraction by 36.

$$\frac{72}{108} = \frac{72 \div 36}{108 \div 36} = \frac{2}{3}$$

A rational expression represents a real number for all real values of the variables involved, except for those values which make the denominator equal zero. Thus, the Principle of Equivalent Fractions can also be used to write rational expressions in lowest terms.

EXAMPLE: Write the rational expression $\frac{5x^3y}{10xz}$ in lowest terms.

Solution

Note that the numerator and denominator of the rational expression $\frac{5x^3y}{10xz}$ have the common factors 5 and x.

First, we can divide both the numerator and denominator by 5.

$$\frac{\overset{1}{\cancel{5}}x^3y}{\underset{2}{\cancel{10}}xz}$$

The numeral 1 is written above the factor 5 to indicate that 5 ÷ 5 = 1. The numeral 2 is placed below the factor 10 to indicate that 10 ÷ 5 = 2.

Next, we can divide both the numerator and denominator by x.

$$\frac{\overset{1}{\cancel{5}}\overset{x^2}{\cancel{x^3}}y}{\underset{2}{\cancel{10}}\underset{1}{\cancel{x}}z}$$

The expression x^2 is placed above the factor x^3 to indicate that $x^3 \div x = x^2$. The numeral 1 is placed below the factor x in the denominator to indicate that $x \div x = 1$.

We can now write the factors that remain in the numerator and denominator.

$$= \frac{1 \times x^2 \times y}{2 \times 1 \times z}$$

$$= \boxed{\frac{x^2y}{2z}}$$

Thus, the rational expression $\frac{5x^3y}{10xz}$ can be written in lowest terms as $\frac{x^2y}{2z}$.

For each rational expression below, state the factors that are common to both the numerator and denominator. Then cancel common factors in order to write the rational expression in lowest terms.

Rational Expression	Common Factors	Cancellation	Lowest Terms
$\frac{-18a^2b^2}{24ab^4}$	6, a, b^2	$\frac{\overset{-3}{\cancel{-18}}\overset{a}{\cancel{a^2}}\overset{1}{\cancel{b^2}}}{\underset{4}{\cancel{24}}\underset{1}{\cancel{a}}\underset{b^2}{\cancel{b^4}}}$	$\frac{-3a}{4b^2}$
$\frac{7x^2}{21xy}$	7, x		
$\frac{-4ab}{16a^2b^3}$			
$\frac{64y^4z^5}{16y^3z^2}$			
$\frac{-9ab^2c^2}{-3ab^2c^3}$	-3,		

Some rational expressions have numerators and denominators that are polynomials of more than one term. In order to write such expressions in lowest terms, you must try to factor the numerator and denominator and cancel any common factors that appear.

EXAMPLE: Write the rational expression $\frac{x^2 + x - 2}{x^2 - 4}$ in lowest terms.

Solution

The trinomial $x^2 + x - 2$ can be written in factored form as $(x + 2)(x - 1)$.

The binomial $x^2 - 4$ can be written in factored form as $(x + 2)(x - 2)$.

Thus,

$$\frac{x^2 + x - 2}{x^2 - 4} = \frac{(x + 2)(x - 1)}{(x + 2)(x - 2)}$$

The common factor $(x + 2)$ can be cancelled in the numerator and denominator.

$$= \frac{\cancel{(x + 2)}(x - 1)}{\cancel{(x + 2)}(x - 2)}$$

$$= \boxed{\frac{x - 1}{x - 2}}$$

Put the following rational expressions in lowest terms.

1. $\frac{x^2 + 3x - 10}{2x^2 - 50} = \frac{(x + 5)(x - 2)}{2(x^2 - 25)} = \frac{\cancel{(x + 5)}(x - 2)}{2\cancel{(x + 5)}(x - 5)} =$

2. $\frac{3}{6x - 12} = \frac{\cancel{3}}{\underset{2}{\cancel{6}}(\_\_ - \_\_)} = \frac{1}{2(\_\_ - \_\_)}$

3. $\frac{y^2 - y - 2}{2y^2 - 5y + 2} = \frac{(y \qquad)(y \qquad)}{(2y \qquad)(y \qquad)} =$

4. $\frac{x^2 - 1}{x^2 - 3x + 2} = \frac{(x \qquad)(x \qquad)}{(x \qquad)(x \qquad)} =$

5. $\frac{3x^2 - 3y^2}{(x + y)^2} = \frac{3(\qquad)}{(x + y)(x + y)} = \frac{3(\qquad)(\qquad)}{(\qquad)(\qquad)} =$

Any rational expression in which the numerator and denominator are identical polynomials is equal to 1.

EXAMPLES:

$$\frac{3x^2y^3}{3x^2y^3} = 1 \qquad \frac{2a - b}{-b + 2a} = 1 \qquad \frac{5x^2 - 5y^2}{5(x - y)(x + y)} = 1$$

Any rational expression in which the numerator and denominator are polynomials that are additive inverses of each other is equal to -1. (Recall that two polynomials are additive inverses if their individual terms are additive inverses.)

EXAMPLES:

$$\frac{x-2}{2-x} = \frac{-1\overset{1}{\cancel{(2-x)}}}{\underset{1}{\cancel{(2-x)}}} = -1 \quad \Bigg\}$$

(x - 2) and (2 - x) are additive inverses since x and -x are opposites and -2 and 2 are opposites.

$$\frac{-a-b}{a+b} = \frac{-1\overset{1}{\cancel{(a+b)}}}{\underset{1}{\cancel{(a+b)}}} = -1 \quad \Bigg\}$$

(-a - b) and (a + b) are additive inverses since -a and a are opposites and -b and b are opposites.

$$\frac{x-5}{-x+5} = \frac{-1\overset{1}{\cancel{(-x+5)}}}{\underset{1}{\cancel{(-x+5)}}} = -1 \quad \Bigg\}$$

(x - 5) and (-x + 5) are additive inverses since x and -x are opposites and -5 and 5 are opposites.

When simplifying rational expressions, you will sometimes find that factors in the numerator and denominator are additive inverses of each other. Such factors can be cancelled to obtain a quotient of -1.

EXAMPLE: Put the rational expression $\frac{4-x^2}{x^2+x-6}$ in lowest terms.

Solution

Factor the numerator and denominator of the rational expression.

$$\frac{4-x^2}{x^2+x-6} = \frac{(2-x)(2+x)}{(x+3)(x-2)}$$

Note that the factor (2 - x) in the numerator and the factor (x - 2) in the denominator are not identical, but are additive inverses of each other. They can be cancelled to give a quotient of -1. (Note that the numeral -1 is placed above the factor (2 - x) to indicate that $\frac{2-x}{x-2} = -1$.)

$$= \frac{\overset{-1}{\cancel{(2-x)}}(2+x)}{(x+3)\underset{1}{\cancel{(x-2)}}}$$

$$= \frac{-(2+x)}{(x+3)}$$

Simplify the following rational expressions. (Cancel factors that are additive inverses of each other to obtain a quotient of -1.)

1. $\frac{-3c - 2d}{9c^2 - 4d^2} = -\frac{\overset{-1}{\cancel{(-3c - 2d)}}}{\underset{1}{\cancel{(3c + 2d)}}(\qquad\qquad)} = \underline{\qquad\qquad}$

2. $\frac{x^2 - 5x + 6}{9 - x^2} = \frac{(x \qquad\quad)(x \qquad\quad)}{(3 \qquad\quad)(3 \qquad\quad)} =$

3. $\frac{(c - d)(3t + 3c)}{(2c + 2t)(d - c)} =$

4. $\frac{2x^2 + 5x - 12}{15 - 10x} = \frac{(2x \qquad\quad)(x \qquad\quad)}{5(\qquad\quad)}$

Self-correcting Exercise #8

Answers may be found on page 66 of this lesson.

1. Fill in the following chart. In Step 1, factor the numerator and denominator if they have more than one term. In Step 2, list the common factors of the numerator and denominator. In Step 3, write the rational expression in lowest terms.

Rational Expression	Step 1	Step 2	Step 3
$\frac{5x^2 + 5x}{10x^3} =$	$\frac{5x(x + 1)}{10x^3}$	5, ___	$\frac{\overset{1}{\cancel{5}}\overset{1}{\cancel{x}}(x + 1)}{\underset{2}{\cancel{10}}\cancel{x^3}x^2} = \underline{\qquad\qquad}$
$\frac{x^2 + x - 20}{3x + 15}$			
$\frac{6x^2 + 7x - 5}{3x^2 + 2x - 5}$			
$\frac{18a^2 + 21a - 60}{27a^2 - 48}$			
$\frac{t^2 - 7t}{t^2 - 8t + 7}$			
$\frac{4ab^2 + 8b^3}{a^2 - ab - 6b^2}$			

2. Write "yes" if the rational expression equals -1 and "no" if it doesn't.

(a) $\frac{a - b}{b - a}$ (b) $\frac{x + y}{x - y}$ (c) $\frac{-c + d}{d - c}$ (d) $\frac{3a + 4b}{-3a - 4b}$

yes ______ ______ ______

(e) $\frac{-r + 2s}{r - 2s}$ (f) $\frac{-a - 5b}{a - 5b}$ (g) $\frac{-x - y}{x + y}$ (h) $\frac{y - 2x}{2x - y}$

______ ______ ______ ______

3. Put the following rational expressions in lowest terms.

(a) $\frac{-36a^2b^2}{-4ab^4c}$

=

(b) $\frac{6x}{2x^3 - 2xy}$

=

(c) $\frac{3m - 3n}{2n - 2m}$

=

(d) $\frac{6 - t}{t^2 - 36}$

=

(e) $\frac{5x^2 - 5y^2}{x^2 - 2xy + y^2}$

=

(f) $\frac{4b - 16a}{16a^2 - b^2}$

=

C. Multiplying Rational Expressions

In Lesson B, you learned that the product of two rational numbers can be determined by finding the product of the numerators and dividing the result by the product of the denominators. Like factors can be cancelled in the numerator and denominator before the multiplication is completed.

EXAMPLE: Find the product of $\frac{2}{5}$ and $\frac{25}{4}$.

Solution

$$\frac{2}{5} \times \frac{25}{4} = \frac{\overset{1}{\cancel{2}} \times \overset{5}{\cancel{25}}}{\underset{1}{\cancel{5}} \times \underset{2}{\cancel{4}}}$$

$$= \frac{1 \times 5}{1 \times 2}$$

$$= \boxed{\frac{5}{2}}$$

Note that the common factors 2 and 5 have been cancelled in the numerator and denominator.

Since rational expressions represent real numbers, they can be multiplied in a similiar manner.

EXAMPLE: Find the product of $\frac{8a}{45c}$ and $\frac{5c^2}{2a^3}$.

Solution

$$\frac{8a}{45c} \times \frac{5c^2}{2a^3} = \frac{\overset{4}{\cancel{8}}\overset{1}{\cancel{a}} \times \overset{1}{\cancel{5}}\overset{c}{\cancel{c^2}}}{\underset{9}{\cancel{45}}\underset{1}{\cancel{c}} \times \underset{1}{\cancel{2}}\underset{a^2}{\cancel{a^3}}}$$

$$= \frac{4 \times c}{9 \times a^2}$$

$$= \boxed{\frac{4c}{9a^2}}$$

Note that the common factors 2, 5, a and c have been cancelled in the numerator and denominator.

Multiply the following rational expressions. Express answers in lowest terms.

1. $\frac{-7}{x} \times \frac{x^2}{14} = \frac{\overset{-1}{\cancel{-7}} \times \overset{x}{\cancel{x^2}}}{\underset{1}{\cancel{x}} \times \underset{2}{\cancel{14}}} = \frac{-1 \times x}{1 \times 2} =$ ____

2. $\frac{-5r}{s} \times \frac{2rs}{-20} = \frac{-5r \times \rule{3em}{0.4pt}}{s \times } =$

3. $\frac{6x^4y}{7a^2b} \times \frac{21a^2b^2}{8xy} =$

4. $$15t \times \frac{3c}{20t^3} = \frac{\overset{3}{\cancel{15}}\overset{1}{\cancel{t}} \times 3c}{\underset{4}{\cancel{20}}\cancel{t^3}\,t^2} =$$

5. $$\frac{-mc}{6a^3} \times \frac{-4a^2}{m^2pc} =$$

Some rational expressions have numerators and/or denominators that have more than one term. In order to multiply such expressions, you must first factor any numerator or denominator that has more than one term. Then any like factors that appear in the numerator and denominator of the product can be cancelled. (Also remember that factors that are additive inverses of each other can be cancelled to give a quotient of -1.)

EXAMPLE: Find the product of $(9 - a^2)$ and $\dfrac{14ax + 7x}{2a^2 - 5a - 3}$.

Solution

First find the product of the numerators and divide this result by the product of the denominators.

$$(9 - a^2) \times \frac{14ax + 7x}{2a^2 - 5a - 3} = \frac{(9 - a^2)(14ax + 7x)}{(2a^2 - 5a - 3)}$$

Then, write the polynomials $(9 - a^2)$, $(14ax + 7x)$, and $(2a^2 - 5a - 3)$ in factored form. Note that $(9 - a^2)$ may be factored as the difference of squares, $(14ax + 7x)$ can be factored by taking the common factor $7x$ out of both terms, and $(2a^2 - 5a - 3)$ can be factored as a trinomial in one variable.

$$= \frac{(3 - a)(3 + a)7x(2a + 1)}{(2a + 1)(a - 3)}$$

The common factor $(2a + 1)$ can be cancelled in both the numerator and denominator. Also the factors $(3 - a)$ and $(a - 3)$ are additive inverses of each other and can be cancelled to give a quotient of -1.

$$= \frac{\overset{-1}{\cancel{(3 - a)}}(3 + a)7x\overset{1}{\cancel{(2a + 1)}}}{\underset{1}{\cancel{(2a + 1)}}\underset{1}{\cancel{(a - 3)}}}$$

$$= \underline{\underline{-7x(3 + a)}}$$

Find the following product.

$$\frac{x^2 - 5x - 14}{x^2 + 6x + 9} \times \frac{x^2 - 9}{3x - 21} = \frac{(x^2 - 5x - 14)(\qquad\qquad)}{(\qquad\qquad)(3x - 21)}$$

$$= \left.\frac{(x - 7)(\qquad)(x - 3)(\qquad)}{(x + 3)(\qquad)\;3\;(\qquad)}\right\}\text{Cancel at this step.}$$

$$= \frac{\qquad\qquad}{3(x + 3)}$$

Self-correcting Exercise #9

Answers may be found on page 67 of this lesson.

1. State any factors that are common to both the numerator and denominator of each product.

Common Factors

(a) $\frac{6x^2}{11y} \times \frac{y}{8x^3}$ $2, x^2, y$

(b) $\frac{-5xy}{9z^3} \times \frac{27z^4}{y^2}$ ______

(c) $\frac{-2x^2y^4}{3z} \times \frac{6z^3}{y^2}$ ______

(d) $\frac{x - 4}{x + 4} \times \frac{x + 4}{x - 4}$ ______

(e) $\frac{6}{y + 5} \times \frac{y^2 - 25}{8}$ ______

2. Write each product in factored form and then state any factors that are common to both the numerator and denominator.

	Factored Form	Common Factors
$\frac{k^2 - k}{k^2 - 4} \times \frac{k^2 + 2k}{k^2 - k}$	$\frac{k(k - 1)\ k(k + 2)}{(k - 2)(k + 2)k(k - 1)}$	k, (k - 1), (k + 2)
$\frac{m^2 + 3m}{m^2 - 9} \times \frac{m^2 - 3m}{m^2}$		
$\frac{4x^2 - 12x + 9}{2x - 3} \times \frac{3x - 5}{4x^2 - 9}$		
$\frac{x^2 - x - 12}{x^2 - 2x - 3} \times \frac{x^2 + 5x + 4}{x^2 - 16}$		
$\frac{3m - 21}{6m + 12} \times \frac{2m^2 + 8m + 8}{m^2 - 49}$		

3. Find the following products. Answers should be expressed in lowest terms.

(a) $\frac{2x}{3y} \times \frac{9y^2}{8x^3}$

(b) $\frac{2x}{5y^2} \times -5y$

(c) $\frac{5z - 5y}{3y + 3z} \times \frac{y + z}{y - z}$

(d) $\frac{x^2 - 1}{x^2 - 9} \times \frac{x^2 + 3x}{x^2 + x}$

(e) $\frac{3k + 2t}{t} \times \frac{t}{-3k - 2t}$

(f) $\frac{c^2 - 5c + 6}{c^2 + 3c} \times \frac{c + 3}{c - 2}$

D. Dividing Rational Expressions

You will recall that the quotient of two rational numbers can be determined by multiplying the first number by the multiplicative inverse of the second number.

EXAMPLE: Divide $\frac{-25}{36}$ by $\frac{-5}{3}$.

Solution

$$\frac{-25}{36} \div \frac{-5}{3} = \frac{-25}{36} \times \frac{3}{-5} = \frac{\overset{5}{\cancel{-25}} \times \overset{1}{\cancel{3}}}{\underset{12}{\cancel{36}} \times \underset{1}{\cancel{-5}}} = \frac{5 \times 1}{12 \times 1} = \frac{5}{12}$$

(Change the operation to multiplication.)

(Give the multiplicative inverse of the divisor.)

Since rational expressions represent real numbers, they can be divided in a similar manner. You must change the operation to multiplication and give the multiplicative inverse of the divisor. Then the rules for multiplying rational expressions can be applied.

EXAMPLE: Divide $\frac{3x^2y}{z^3}$ by $\frac{12xy^3}{5z^5}$.

Solution

$$\frac{3x^2y}{z^3} \div \frac{12xy^3}{5z^5}$$

Dividing by $\frac{12xy^3}{5z^5}$ is equivalent to multiplying by $\frac{5z^5}{12xy^3}$.

$$= \frac{3x^2y}{z^3} \times \frac{5z^5}{12xy^3}$$

Follow the instructions on page 40 of this lesson for multiplying rational expressions.

$$= \frac{\overset{1}{\cancel{3}}\overset{x}{\cancel{x^2}}\overset{1}{\cancel{y}} \times 5\overset{z^2}{\cancel{z^5}}}{\underset{1}{\cancel{z^3}} \times \underset{4}{\cancel{12}}\underset{1}{\cancel{x}}\underset{y^2}{\cancel{y^3}}}$$

$$= \frac{x \times 5z^2}{1 \times 4y^2}$$

$$= \boxed{\frac{5xz^2}{4y^2}}$$

Find the following quotients.

1. $\frac{8a^2}{b} \div \frac{12a}{b} = \frac{8a^2}{b} \times \frac{b}{12a} = \frac{\overset{2}{\cancel{8}}\overset{a}{\cancel{a^2}} \times \overset{1}{\cancel{b}}}{\underset{1}{\cancel{b}} \times \underset{3}{\cancel{12}}\underset{1}{\cancel{a}}} = \frac{\qquad \times 1}{1 \times \qquad} =$

2. $\frac{-5x^2y^3}{z} \div \frac{15xy^4}{z^4} = \frac{-5x^2y^3}{z} \times \frac{\qquad}{15xy^4} = \frac{-\overset{-1}{\cancel{5}}\overset{x}{\cancel{x^2}}\overset{1}{\cancel{y^3}} \times \qquad}{z \times \underset{3}{\cancel{15}}\underset{1}{\cancel{x}}\underset{y}{\cancel{y^4}}} =$

3. $\frac{xy}{6z^2} \div \frac{y^3}{14z} = \underline{\qquad\qquad} \times \frac{14z}{\qquad} =$

4. $\frac{5a^2b}{c^3} \div 10a^2b = \underline{\qquad} \times \frac{1}{\qquad} =$

If the numerators and/or denominators of the rational expressions you are dividing have more than one term, proceed in the same manner. Change the operation to multiplication and give the reciprocal of the divisor. Then multiply as you were shown on page 41 of this lesson. (Factor numerators and denominators where possible.)

EXAMPLE: Divide $\frac{3x + x^2}{x + 3x^2}$ by $\frac{9 - x^2}{1 + 3x}$.

Solution

$$\frac{3x + x^2}{x + 3x^2} \div \frac{9 - x^2}{1 + 3x}$$

Dividing by $\frac{9 - x^2}{1 + 3x}$ is equivalent to multiplying by $\frac{1 + 3x}{9 - x^2}$.

$$= \frac{3x + x^2}{x + 3x^2} \times \frac{1 + 3x}{9 - x^2}$$

Find the product of the numerators and divide this result by the product of the denominators.

$$= \frac{(3x + x^2)(1 + 3x)}{(x + 3x^2)(9 - x^2)}$$

The binomials $(3x + x^2)$ and $(x + 3x^2)$ can be factored by taking the common factor x out of the two terms. The binomial $(9 - x^2)$ can be factored as the difference of two squares.

$$= \frac{\cancel{x}\cancel{(3 + x)}\cancel{(1 + 3x)}}{\cancel{x}\cancel{(1 + 3x)}\cancel{(3 + x)}(3 - x)} \quad \Bigg\}\ \text{Note how like factors have been cancelled.}$$

$$= \boxed{\frac{1}{3 - x}}$$

Find the following quotient.

$$\frac{a^2 + 4a - 12}{a^2 + 9a + 18} \div \frac{3a + 12}{6a + 18} = \frac{a^2 + 4a - 12}{a^2 + 9a + 18} \times \frac{\qquad\qquad}{3a + 12}$$

$$= \frac{(a^2 + 4a - 12)(\qquad\qquad)}{(\qquad\qquad)(\qquad\qquad)}$$

Cancel like factors. → $= \frac{(a \qquad)(a \qquad)\ 6\ (a \qquad)}{(a \qquad)(a \qquad)\ 3\ (a \qquad)}$

$$= \frac{\qquad\qquad}{(a + 4)}$$

Self-correcting Exercise #10

Answers may be found on page 68 of this lesson.

1. Find the following quotients. Answers should be expressed in lowest terms.

(a) $\frac{5a^2c^2}{b^2} \div \frac{-20ac}{3b^3}$

(b) $\frac{m^2 - n^2}{mn} \div (m - n)$

(c) $\frac{-xy^2}{1 - x} \div \frac{2y}{x - 1}$

(d) $\frac{x^2 - xy}{x^2 + xy} \div \frac{x^2 - y^2}{x + y}$

E. Adding and Subtracting Rational Expressions

You will recall that the sum (or difference) of two rational numbers with the same denominators can be determined by adding (or subtracting) the two numerators and dividing this result by the common denominator.

EXAMPLE 1:

$$\frac{-2}{7} + \frac{3}{7} = \frac{-2 + 3 \leftarrow \text{SUM OF NUMERATORS}}{7 \leftarrow \text{COMMON DENOMINATOR}}$$

$$= \boxed{\frac{1}{7}}$$

EXAMPLE 2:

$$\frac{1}{8} - \frac{-3}{8} = \frac{1-(-3)\ \leftarrow \text{DIFFERENCE OF NUMERATORS}}{8\ \leftarrow \text{COMMON DENOMINATOR}}$$

$$= \frac{1+3}{8}$$

$$= \frac{4}{8}$$

$$= \boxed{\frac{1}{2}}$$

Rational expressions with like denominators can be added and subtracted in a similar manner.

EXAMPLE 1:

$$\frac{-x}{x+y} + \frac{3x}{x+y} = \frac{-x+3x\ \leftarrow \text{SUM OF NUMERATORS}}{x+y\ \leftarrow \text{COMMON DENOMINATOR}}$$

$$= \boxed{\frac{2x}{x+y}}$$

EXAMPLE 2:

$$\frac{2a+b}{c^2} - \frac{a-3b}{c^2} = \frac{(2a+b)-(a-3b)\ \leftarrow \text{DIFFERENCE OF NUMERATORS}}{c^2\ \leftarrow \text{COMMON DENOMINATOR}}$$

When you subtract the two binomials in the numerator, you must change the operation to addition and change the signs of both terms in the second binomial. (Review page 24, lesson D.)

$$= \frac{(2a+b) \overset{\text{ADD}}{+} \overbrace{(-a+3b)}^{\text{ADDITIVE INVERSE OF }(a-3b)}}{c^2}$$

$$= \boxed{\frac{a+4b}{c^2}}$$

Find the following sums and differences.

1. $\frac{2}{m+n} - \frac{n+2}{m+n} = \frac{2-(\qquad)}{m+n} = \frac{2+(-n-2)}{m+n} = \frac{\qquad}{m+n}$

2. $\frac{3}{b} - \frac{a}{b} + \frac{a-b}{b} = \frac{3-a+(\qquad)}{b} = \frac{\qquad}{b}$

3. $\frac{2x}{x^2-16} - \frac{8}{x^2-16} = \frac{\qquad}{x^2-16} = \frac{2(\qquad)}{(\qquad)(\qquad)} = \frac{\qquad}{x+4}$

4. $\frac{x+3}{x} - \frac{2x-4}{x} = \frac{(\qquad)-(\qquad)}{x} = \frac{(\qquad)+(\qquad)}{x} = \frac{\qquad}{x}$

In Lesson B, you were told that the sum (or difference) of two rational numbers with unlike denominators can be determined by first expressing the numbers in terms of their L.C.D. and then applying the rule for adding (or subtracting) two rational numbers with like denominators.

EXAMPLE: Find the sum of $\frac{3}{8}$ and $\frac{5}{36}$.

Solution

The L.C.D. of the denominators 8 and 36 is 72. (This is the smallest number into which both denominators will divide.) We can express $\frac{3}{8}$ with a denominator of 72 by multiplying both its terms by 9. We can express $\frac{5}{36}$ with a denominator of 72 by multiplying both its terms by 2.

i.e. $\frac{3}{8} + \frac{5}{36} = \frac{3 \times 9}{8 \times 9} + \frac{5 \times 2}{36 \times 2}$

Now we can add the numerators and put the result over the common denominator, 72.

$$= \frac{27 + 10}{72}$$

$$= \boxed{\frac{37}{72}}$$

Rational expressions with unlike denominators can be added or subtracted in a similar manner. In order to find the L.C.D. of a number of rational expressions, you must look at the variable factors in the denominator and write each factor the greatest number of times it occurs in any one denominator.

EXAMPLE: Simplify the expression $\frac{3z}{xy^3} + \frac{2}{x^2yz} - \frac{xz + 3}{x^2y^2}$

Solution

Find the L.C.D. by looking at the three denominators and determining the highest power of each variable that occurs in any one denominator. The highest power of x that occurs is x^2, the highest power of y that occurs is y^3, and the highest power of z that occurs is z. Thus, the L.C.D. is x^2y^3z. Each fraction must be changed to this L.C.D.

$\frac{3z}{xy^3}$ can be changed to the L.C.D. by multiplying both terms by xz.

$\frac{2}{x^2yz}$ can be changed to the L.C.D. by multiplying both terms by y^2.

$\frac{xz + 3}{x^2y^2}$ can be changed to the L.C.D. by multiplying both terms by yz.

Thus,

$$\frac{3z}{xy^3} + \frac{2}{x^2yz} - \frac{xz+3}{x^2y^2} = \frac{3z \times xz}{xy^3 \times xz} + \frac{2 \times y^2}{x^2yz \times y^2} - \frac{(xz+3) \times yz}{x^2y^2 \quad yz}$$

MULTIPLY BOTH TERMS BY xy. MULTIPLY BOTH TERMS BY y^2. MULTIPLY BOTH TERMS BY yz.

Now we can combine the numerators and put the result over the common denominator, x^2y^3z.

$$= \frac{(3z \times xz) + (2 \times y^2) - (xz+3)yz}{x^2y^3z}$$

$$= \frac{3xz^2 + 2y^2 - (xyz^2 + 3yz)}{x^2y^3z}$$

$$= \frac{3xz^2 + 2y^2 - xyz^2 - 3yz}{x^2y^3z}$$

Find the following sums and differences.

1. $\frac{1}{ab} + \frac{2}{bc} = \frac{1 \times c}{ab \times c} + \frac{2 \times a}{bc \times a} = \frac{(1 \times c) + (2 \times a)}{abc} = \frac{\qquad + \qquad}{abc}$
   (L.C.D. is abc.)

2. $\frac{2}{x} - \frac{5}{xy} - \frac{3}{y} = \frac{2 \times y}{x \times y} - \frac{5}{xy} - \frac{3 \times x}{y \times x} = \frac{\qquad\qquad - 3x}{xy}$
   (L.C.D. is xy.)

3. $\frac{2}{xy} + \frac{3}{x^2} + \frac{4}{yz} = \frac{2 \times xz}{xy \times xz} + \frac{3 \times \quad}{x^2 \times yz} + \frac{4 \times x^2}{yz \times \quad} = \frac{2xz + \qquad\qquad}{\qquad}$
   (L.C.D. is x^2yz.)

4. $\frac{5}{r} - \frac{3}{t} = \frac{5 \times \qquad}{r \times \qquad} - \frac{3 \times \qquad}{t \times \qquad} = \frac{\qquad\qquad}{rt}$
   (L.C.D. is ______.)

5. $\frac{2}{x} - \frac{x-3}{2x} = \frac{2 \times \qquad}{x \times \qquad} - \frac{x-3}{2x} = \frac{4 - (\qquad)}{\qquad} = \frac{4 + (\qquad)}{2x} = \frac{\qquad\qquad}{2x}$
   (L.C.D. is ______.)

In some cases when you are adding or subtracting rational expressions, the denominators will be polynomials of more than one term. You must try to factor these denominators before you try to determine the L.C.D. The L.C.D. will contain each factor from the denominators the greatest number of times its occurs in any one denominator.

EXAMPLE: Simplify the expression $\frac{2}{x^2 + 2xy + y^2} + \frac{3}{x^2 - 2xy + y^2} - \frac{1}{x^2 - y^2}$.

Solution

We can determine the L.C.D. by factoring the three denominators and picking out each factor the greatest number of times that it occurs in any one denominator. Note that:

$$\begin{aligned} x^2 + 2xy + y^2 &= (x + y)^2 \\ x^2 - 2xy + y^2 &= (x - y)^2 \\ x^2 - y^2 &= (x + y)(x - y) \end{aligned}$$

Since the factor $(x + y)$ appears in the first denominator twice, it must appear in the L.C.D. twice.

Since the factor $(x - y)$ appears in the second denominator twice, it must appear in the L.C.D. twice.

$\therefore$ L.C.D. $= (x + y)^2(x - y)^2$

Note how the three fractions can be changed to the L.C.D.

1. Both terms of $\frac{2}{(x + y)^2}$ must be multiplied by $(x - y)^2$.
2. Both terms of $\frac{3}{(x - y)^2}$ must be multiplied by $(x + y)^2$.
3. Both terms of $\frac{1}{(x + y)(x - y)}$ must be multiplied by $(x + y)(x - y)$.

Write down your solution to this question in the following manner.

$$\frac{2}{x^2 + 2xy + y^2} + \frac{3}{x^2 - 2xy + y^2} - \frac{1}{x^2 - y^2}$$

Factor denominators.

$$= \frac{2}{(x + y)^2} + \frac{3}{(x - y)^2} - \frac{1}{(x + y)(x - y)}$$

Change each fraction to the L.C.D.

$$= \frac{2}{(x + y)^2}\underset{\substack{\uparrow \\ \text{MULTIPLY BOTH} \\ \text{TERMS BY } (x-y)^2.}}{\boxed{\frac{(x - y)^2}{(x - y)^2}}} + \frac{3}{(x - y)^2}\underset{\substack{\uparrow \\ \text{MULTIPLY BOTH} \\ \text{TERMS BY } (x+y)^2.}}{\boxed{\frac{(x + y)^2}{(x + y)^2}}} - \frac{1}{(x + y)(x - y)}\underset{\substack{\uparrow \\ \text{MULTIPLY BOTH TERMS} \\ \text{BY } (x+y)(x-y).}}{\boxed{\frac{(x + y)(x - y)}{(x + y)(x - y)}}}$$

Combine numerators and place result over the common denominator.

$$= \frac{2(x - y)^2 + 3(x + y)^2 - (x + y)(x - y)}{(x + y)^2(x - y)^2}$$

Simplify the numerator.

$$= \frac{2(x^2 - 2xy + y^2) + 3(x^2 + 2xy + y^2) - (x^2 - y^2)}{(x + y)^2(x - y)^2}$$

$$= \frac{2x^2 - 4xy + 2y^2 + 3x^2 + 6xy + 3y^2 - x^2 + y^2}{(x + y)^2(x - y)^2}$$

$$= \boxed{\frac{4x^2 + 2xy + 6y^2}{(x + y)^2(x - y)^2}}$$

Simplify the following expression.

$$\frac{x + y}{4xy} - \frac{y}{6x^2 - 6xy} + \frac{x}{8xy - 8y^2}$$

$$= \frac{x + y}{4xy} - \frac{y}{6x(\qquad)} + \frac{x}{8y(\qquad)}$$

The L.C.D. is 24xy(x - y).

$$= \frac{(x + y)\ 6(x - y)}{(4xy)\ 6(x - y)} - \frac{y(\quad)}{6x(\qquad)(4y)} + \frac{x(3x)}{8y(\qquad)(\quad)}$$

$$= \frac{6(x + y)(\qquad) - y(\qquad) + x(3x)}{24xy(x - y)}$$

$$= \frac{6(\qquad) - \qquad + \qquad}{24xy(x - y)}$$

$$= \frac{6x^2 \qquad\qquad}{24xy(x - y)}$$

$$= \frac{\qquad\qquad}{24xy(x - y)}$$

Sometimes when you are adding or subtracting rational expressions, you will find that some of the factors of the denominators are additive inverses. Your work will be simplified if you find a way to make these factors identical.

EXAMPLE: Find the sum of $\frac{3x^2}{x^2 - y^2}$ and $\frac{x - 2y}{y - x}$.

Solution

$$\frac{3x^2}{x^2 - y^2} + \frac{x - 2y}{y - x} = \frac{3x^2}{(x - y)(x + y)} + \frac{x - 2y}{(y - x)}$$

Note that the factors $(x - y)$ and $(y - x)$ are not identical, but are additive inverses of each other. The second fraction can be made to have a denominator of $(x - y)$ if both of its terms are multiplied by -1.

$$= \frac{3x^2}{(x - y)(x + y)} + \frac{(-1)(x - 2y)}{(-1)(y - x)}$$

$$= \frac{3x^2}{(x - y)(x + y)} + \frac{-x + 2y}{(x - y)}$$

The L.C.D. is $(x - y)(x + y)$.

$$= \frac{3x^2}{(x - y)(x + y)} + \frac{(-x + 2y)\boxed{(x + y)}}{(x - y)\boxed{(x + y)}}$$

MULTIPLY BOTH TERMS BY $(x+y)$.

$$= \frac{3x^2 + (-x + 2y)(x + y)}{(x - y)(x + y)}$$

$$= \frac{3x^2 + (-x^2) + xy + 2y^2}{(x - y)(x + y)} = \boxed{\frac{2x^2 + xy + 2y^2}{(x - y)(x + y)}}$$

Find the following difference.

$$\frac{2x + 1}{x^2 - x} - \frac{x - 5}{1 - x} = \frac{2x + 1}{x(\qquad)} - \frac{x - 5}{1 - x}$$

$$= \frac{2x + 1}{x(\qquad)} - \frac{\boxed{(-1)}(x - 5)}{\boxed{(-1)}(1 - x)}$$

MULTIPLY BOTH TERMS BY -1.

$$= \frac{2x + 1}{x(\qquad)} - \frac{-x + \quad}{x - 1}$$

L.C.D. is $x(x - 1)$.

$$= \frac{2x + 1}{x(\qquad)} - \frac{(-x + \quad)\boxed{x}}{(x - 1)\boxed{x}}$$

MULTIPLY BOTH TERMS BY x.

$$= \frac{(\qquad) - (-x + \quad)x}{x(\qquad)}$$

$$= \frac{(\qquad) - (-x^2 + \quad)}{x(\qquad)}$$

$$= \frac{(\quad + 1) + (\quad -5x)}{x(\qquad)}$$

$$= \frac{\qquad\qquad}{x(\qquad)}$$

Self-correcting Exercise #11

Answers may be found on page 69 of this lesson.

1. State the L.C.D. of each of the following sets of rational expressions.

L.C.D.

(a) $\frac{3}{mn}, \frac{5}{mn^2}, \frac{12}{m^2n^2}$ ____________

(b) $\frac{x}{bc}, \frac{x + c}{ac}, \frac{x + b}{ab}$ ____________

(c) $\frac{x^2}{(x + y)(x + y)}, \frac{y^2}{(x + y)(x - y)}$ ____________

(d) $\frac{y}{x(x + y)}, \frac{x}{y(x + y)}$ ____________

(e) $\frac{2a + 3}{3a^2b}, \frac{a + 2b}{4ab^2}, \frac{1}{6ab}$ ____________

(f) $\frac{3t + 1}{t(t - 1)^2}, \frac{2t - 3}{(t - 1)(t + 1)^2}$ ____________

2. Change the rational expressions in each question to their L.C.D.

(a) $\frac{a}{4}, \frac{b}{6}, \frac{c}{8}$

L.C.D. is ____________.

$\frac{a}{4} =$

$\frac{b}{6} =$

$\frac{c}{8} =$

(b) $\frac{z}{xy}, \frac{x}{yz}, \frac{y}{zx}$

L.C.D. is ____________.

$\frac{z}{xy} =$

$\frac{x}{yz} =$

$\frac{y}{zx} =$

(c) $\frac{x + 2}{x^2 - 7x + 12}, \frac{x + 4}{x^2 - 9}$

$x^2 - 7x + 12 = (\quad)(\quad)$

$x^2 - 9 = (\quad)(\quad)$

L.C.D. is ____________.

$\frac{x + 2}{(\quad)(\quad)} =$

$\frac{x + 4}{(\quad)(\quad)} =$

(d) $\frac{x}{x^2 - 25}, \frac{1}{2x + 10}$

$x^2 - 25 = (\quad)(\quad)$

$2x + 10 = \_\_\_(\quad)$

L.C.D. is ____________.

$\frac{x}{(\quad)(\quad)} =$

$\frac{1}{2(\quad)} =$

3. Simplify the following expressions. Answers should be expressed in lowest terms.

(a) $\frac{3y}{z} - \frac{2y - 7}{z} + \frac{5}{z}$

$=$

(b) $\frac{-3x + y}{x - y} + \boxed{\frac{2y - x}{y - x}}$ MULTIPLY BOTH TERMS BY −1.

$=$

(c) $\frac{a + b}{15} + \frac{a - b}{20} + \frac{2b}{4}$

$=$

(d) $\frac{3}{x^2} + \frac{5}{x} - 2$

$=$

(e) $\frac{y^2 - 2}{y^2 - y - 2} - \frac{y + 1}{y - 2} =$

(f) $\frac{2a}{a^2 - 9} + \frac{1}{a - 3} - \frac{1}{a + 3} =$

EXERCISE - Rational Expressions

1. Fill in the blanks.

(a) A fraction must not have a denominator whose value is ______.

(b) (2 - a) is equal to the product of -1 and __________.

(c) A rational expression is any expression that can be written as the ________________ of two ____________________. (See p. 33.)

(d) (a - b) and (b - a) are ________________ inverses of each other.

(e) The quotient $\frac{5}{x} \div y$ can be determined by multiplying $\frac{5}{x}$ by ______.

(f) The difference $\frac{a}{c} - \frac{b}{c}$ can be determined by subtracting the ____________________ and putting the result over the common ____________________. (i.e. $\frac{a}{c} - \frac{b}{c} = \overline{\qquad}$)

(g) All polynomials are rational expressions with denominators of _____.

(h) A rational expression is in simplest form when its numerator and denominator have no common ____________________.

(i) When multiplying rational expressions, we can ____________________ any like factors that appear in both the numerator and denominator of the product.

(j) Any rational expression that equals -1 has a numerator and denominator that are _____________ ____________ of each other.

2. Fill in the missing numerators.

(a) $\frac{a + b}{12} = \frac{a(a + b)}{12a}$ BOTH TERMS HAVE BEEN MULTIPLIED BY "a".

(b) $\frac{3}{10} = \frac{\quad}{20ab}$ } MULTIPLY BOTH TERMS BY 2ab.

(c) $\frac{a}{b} = \frac{\quad}{b^2}$

(d) $\frac{3k}{5} = \frac{\quad}{30k}$

(e) $x^2 = \frac{\quad}{y}$

(f) $\frac{5}{x + y} = \frac{\quad}{(x + y)^2}$

3. State the values of the variable for which each of the following rational expressions is not defined.

(a) $\frac{-6}{a}$

(b) $\frac{a + b}{a - 3}$

(c) $\frac{2x}{x^2 - 25}$

(HINT: Factor the denominator.)

(d) $\frac{5}{x^2 + 2x}$

(e) $\frac{ab}{(a + 5)(a - 2)}$

(f) $\frac{x + 2}{2x^2 - 10x + 12}$

4. Write the following rational expressions in lowest terms. Some of these expressions are already in lowest terms and cannot be simplified further. Under these expressions write "in lowest terms."

CAUTION: When simplifying, cancel factors but not terms. Review definitions of "factors" and "terms" on pages 32 and 33, lesson A.

(a) $\frac{13a^2bc}{26ab^3}$

(b) $\frac{cy + cz}{2y + 2z}$

$= \frac{c(\qquad)}{2(\qquad)}$

$=$

(c) $\frac{4 + x}{4 + y}$

(in lowest terms)

NOTE: The 4's can't be cancelled here since they are terms - not factors.

(d) $\frac{x - 4}{x^2 - 4}$

(e) $\frac{5xy^2}{10y}$

(f) $\frac{5r + 5s}{5rs}$

(g) $\frac{a + b}{2ab}$

(h) $\frac{2a - 3b}{3b - 2a}$

(i) $\frac{(3x + 2)^2}{9x^2 - 4}$

(j) $\frac{(x + y)^2}{x + y}$

(k) $\frac{3a + 2b}{3a}$

(l) $\frac{8n^2 + 10n - 3}{8n^2 - 14n + 3}$

(m) $\frac{x^2 - 2x - 3}{x^2 + 2x - 15}$

(n) $\frac{x^2 + 1}{x + 1}$

(o) $\frac{r^2 - 6r + 9}{9 - r^2}$

(p) $\frac{3m}{3 + m}$

5. Decide whether each of the following statements is true or false.

True or False?

(a) The L.C.D. of $\frac{3t + 1}{(t - 1)^2}$ and $\frac{2t - 3}{t^2 - 1}$ is $(t + 1)(t - 1)$. false

(b) $\frac{1}{3} + \frac{1}{4} = \frac{1}{7}$ ________

(c) $m \times \frac{a}{b} = \frac{ma}{mb}$ ________

(d) $\frac{s - t}{t - s} = -1$ ________

(e) $\frac{3}{p + 3} = \frac{1}{p + 1}$ ________

(f) The L.C.D. of $\frac{1}{x^2}$ and $\frac{5}{xy}$ is x^2y. ________

(g) $\frac{a - a}{t^2} = 0$ ________

(h) $\frac{a + b}{b + a} = 0$ ________

(i) $\frac{n - 1}{d - 1} = \frac{n}{d}$ ________

(j) $\frac{1}{2y} + \frac{3}{y^2} = \frac{y + 6}{2y^2}$ ________

(k) $\frac{4x - (y + 3)}{5} = \frac{4x - y + 3}{5}$ ________

(l) The L.C.D. of $\frac{1}{x}$, $\frac{2}{x^2y}$, and $\frac{3}{xyz^3}$ is xyz^3. ________

True or False?

(m) $\frac{x}{b - a} = \frac{-x}{a - b}$ ________________

(n) $\frac{-x^5}{x^5} = -1$ ________________

(o) $\frac{a}{b} - \frac{b}{a} = \frac{a - b}{ab}$ ________________

(p) $\frac{a}{x} + \frac{b}{x} = \frac{a + b}{2x}$ ________________

(q) $\frac{a}{b} \div 3 = \frac{a}{3b}$ ________________

6. Perform the indicated operations. Express each answer in lowest terms.

(a) $\frac{2z^4}{15x^3y^2} \div \frac{8x}{5y}$

(b) $\frac{7ab}{5cd} \times \frac{15c^2d^2}{28a^2b^2}$

(c) $\frac{4a^2}{7} \div 8a$

(d) $-15x^2y \times \frac{1}{3x^3y^2}$

(e) $\dfrac{a^2 + a - 6}{3} \times \dfrac{1}{a^2 - 4}$

(f) $\dfrac{2x^2 - x}{4x^2 - 4x + 1} \div \dfrac{x^2}{8x - 4}$

(Remember to change the operation to multiplication and give the reciprocal of the divisor.)

(g) $\dfrac{m^2 - 25n^2}{m^2 - 10mn + 25n^2} \times \dfrac{2m - 10n}{m + 5n}$

=

(h) $\dfrac{3n^2 + 7n + 2}{1 - 9n^2} \times \dfrac{3n^2 - 7n + 2}{4 - n^2}$

=

(i) $\dfrac{r^3 - rs^2}{r^2s} \div \dfrac{r^2 - rs}{3rs + 3s^2}$

=

(j) $\dfrac{2x^2 + 7x + 3}{x^2 - 9} \div \dfrac{2x^2 + 11x + 5}{x^2 - 3x}$

=

(k) $\frac{c + d}{c} - \frac{c - d}{c}$

=

(l) $\frac{r - 2}{2r} + \frac{3r + 2}{2r} - \frac{r + 5}{2r}$

=

In doing parts (m) to (q), don't be tempted to cancel. Once each expression is written over a common denominator, the numerator is composed of terms - not factors. Only factors can be cancelled. Review examples on pages 48 - 52.

(m) $\frac{2}{a} + \frac{3}{b} - \frac{1}{c} =$

(n) $\frac{4x}{x^2 - 36} + \frac{3}{x - 6} =$

(o) $\frac{a}{a + b} + \frac{b}{a - b} - \frac{1}{a^2 - b^2} =$

(p) $\frac{y + 1}{y + 2} - \frac{2}{y + 4} + \frac{4}{y^2 + 6y + 8} =$

(q) $\frac{x + 2}{x^2 - 25} - \frac{1}{2x + 10} + \frac{x - 1}{x^2 + 10x + 25}$ HINT: L.C.D. is $2(x + 5)^2(x - 5)$

$=$

Key to Self-correcting Exercises in Lesson E

Exercise #1, page 3

1. (a) 7 (b) a^2 (c) xy (d) t (e) $9x$ (f) $7a^2$
   (g) x^2y^3 (h) b^2c (i) 11

2.

$27xy^3,\ 15x^2y^2$	3	x	y^2	——	$3xy^2$
$8x^3,\ 12x^5$	4	x^3	——	——	$4x^3$
$6y^2z^2,\ 2y^3z$	2	——	y^2	z	$2y^2z$
$24x^4y^3,\ 36x^3y^4$	12	x^3	y^3	——	$12x^3y^3$
$12x^3y^2z^3,\ 6x^2y^3z$	6	x^2	y^2	z	$6x^2y^2z$
$16x^2y^2z^2,\ 12yz^3$	4	——	y	z^2	$4yz^2$
$5x^3yz^3,\ 7x^4y^2z^2,\ 9x^2y^3z^3$	——	x^2	y	z^2	x^2yz^2

Exercise #2, page 5

1. (a) xy (b) $3ab$ (c) $5abc^2$ (d) $7d^2e$ (e) $3mn^2p$
   (f) $-3r^3s$ (g) -1 (h) $-13u^2vw^2$ (i) $2x$ (j) 1

2. (a) $2a$ (b) $9b$ (c) 6 (d) mnt (e) $3y^3$
   (f) $16xy^2$ (g) 3 (h) b (i) $5x$ (j) y
   (k) $9a^3$ (l) $(a + b)$

3.

			Factored Form
$b^3 - b^2 + b$	b	$b(b^2) + b(-b) + b(1)$	$b(b^2 - b + 1)$
$12y^4 + 30y^3 - 42y$	$6y$	$(6y)(2y^3) + (6y)(5y^2) + 6y(-7)$	$6y(2y^3 + 5y^2 - 7)$
$10a^2b - 15ab^2$	$5ab$	$(5ab)(2a) + (5ab)(-3b)$	$5ab(2a - 3b)$
$x^2 + 7x$	x	$x(x) + x(7)$	$x(x + 7)$
$3x^2 - 21x^3$	$3x^2$	$(3x^2)(1) + (3x^2)(-7x)$	$3x^2(1 - 7x)$
$12x^4 - 8x^2 + 20x$	$4x$	$(4x)(3x^3) + (4x)(-2x) + (4x)(5)$	$4x(3x^3 - 2x + 5)$
$5r^2t - 10rt + 5rt^2$	$5rt$	$(5rt)(r) + (5rt)(-2) + (5rt)(t)$	$5rt(r - 2 + t)$

4. (a) $9b^2 - 81b = (9b)(b) + (9b)(-9) = \underline{9b(b - 9)}$
   (b) $9y^5 - 6y^4 + 3y^3 = (3y^3)(3y^2) + (3y^3)(-2y) + (3y^3)(1) = \underline{3y^3(3y^2 - 2y + 1)}$
   (c) $12a^2b^2 + 6ab^2 - 3ab = (3ab)(4ab) + (3ab)(2b) + (3ab)(-1) = \underline{3ab(4ab + 2b - 1)}$
   (d) $3ab - 5b + 7b^2 = b(3a) + b(-5) + b(7b) = \underline{b(3a - 5 + 7b)}$

Exercise #3, page 9

1. (a) $(9x)^2 - (10y)^2$ (b) $(ab)^2 - (1)^2$ (c) $(8a)^2 - (b)^2$

(d) $(1)^2 - (4y^2)^2$ (e) $(3a^4)^2 - (5b^2)^2$ (f) $(x^2y^3)^2 - (15z^5)^2$

2. (a) $5ab[(2a)^2 - b^2] = \underline{5ab(2a - b)(2a + b)}$

(b) $m(x^4 - y^4) = m[(x^2)^2 - (y^2)^2] = m\overbrace{(x^2 - y^2)}^{\text{DIFFERENCE OF SQUARES}}(x^2 + y^2) = \underline{m(x - y)(x + y)(x^2 + y^2)}$

(c) $(4s)^2 - (t)^2 = (4s - t)(4s + t)$

(d) $(4m^2n)^2 - (p^3)^2 = \underline{(4m^2n - p^3)(4m^2n + p^3)}$

(e) $a(y^4 - 1) = a[(y^2)^2 - (1)^2] = a(y^2 - 1)(y^2 + 1) = \underline{a(y - 1)(y + 1)(y^2 + 1)}$

(f) $2(25x^4 - 16y^4) = 2[(5x^2)^2 - (4y^2)^2] = \underline{2(5x^2 - 4y^2)(5x^2 + 4y^2)}$

(g) $\underline{(x^2 + 1 - x)(x^2 + 1 + x)}$

(h) $[a - (b + c)]\,[a + (b + c)] = \underline{(a - b - c)(a + b + c)}$

Exercise #4, page 13

1. (a) yes; $(x + 4)^2$ (b) yes; $(p - 1)^2$ (c) no

(d) no (e) yes; $(r - s)^2$ (f) yes; $(3t + 5)^2$

(g) no (h) yes; $(2x^2 + 3y^2)^2$ (i) no

2. (a) $a^2 + \underline{10a} + 25 = \underline{(a + 5)^2}$ (b) $16x^2 + \underline{24x} + 9 = \underline{(4x + 3)^2}$

(c) $9y^2 - \underline{6y} + 1 = \underline{(3y - 1)^2}$ (d) $4x^2 + 12x + \underline{9} = \underline{(2x + 3)^2}$

(e) $\underline{x^2} - 16x + 64 = \underline{(x - 8)^2}$ (f) $9a^2 - 12ab + \underline{4b^2} = \underline{(3a - 2b)^2}$

(g) $\underline{9x^2} - 30x + 25 = \underline{(3x - 5)^2}$ (h) $16x^2y^2 - \underline{8xy} + 1 = \underline{(4xy - 1)^2}$

3. (a) $\underline{2n(n + 2)^2}$

(b) $3(9x^2 - 6x + 1) = \underline{3(3x - 1)^2}$

(c) $(b)^2 + 2(b)(7) + (7)^2 = \underline{(b + 7)^2}$

(d) $(4x)^2 + 2(4x)(-1) + (-1)^2 = \underline{(4x - 1)^2}$

(e) $(12n)^2 + 2(12n)(5x) + (5x)^2 = \underline{(12n + 5x)^2}$

(f) $3(81m^2n^2 + 18mn + 1) = 3[(9mn)^2 + 2(9mn)(1) + (1)^2] = \underline{3(9mn + 1)^2}$

(g) $(4x)^2 + 2(4x)(-3y) + (-3y)^2 = \underline{(4x - 3y)^2}$

(h) $(x^2)^2 + 2(x^2)(y^2) + (y^2)^2 = \underline{(x^2 + y^2)^2}$

Exercise #5, page 18

1. (a) 6 and 4 (b) -6 and -4 (c) 2 and 9
   (d) 10 and 5 (e) 4 and 11 (f) 4 and -1
   (g) -1 and -3 (h) 8 and -12 (i) -4 and -15

2. (a) $(x - 6)(x + 2)$ (b) $(x + 7)(x + 5)$ (c) $(x - 9)(x + 7)$
   (d) $(x + 4)(x - 2)$ (e) $(x - 15)(x + 2)$ (f) $(x - 8)(x + 5)$
   (g) $(x + 3)(x + 5)$ (h) $(x - 8)(x + 7)$ (i) $-(x + 7)(x - 3)$
   (j) $-(x^2 - 6x - 27)$ $= -(x - 9)(x + 3)$
   (k) $3(x^2 + 3x - 10)$ $= 3(x + 5)(x - 2)$
   (l) $-2(x^2 - x - 2)$ $= -2(x - 2)(x + 1)$

Exercise #6, page 25

1. (a) $6x^2 + 17x + 5 = \underline{(2x + 5)}(3x + 1)$
   (b) $2y^2 + 13y + 15 = (2y + 3)\underline{(y + 5)}$
   (c) $12a^2 + 28a + 15 = \underline{(6a + 5)}(2a + 3)$
   (d) $x^2 + 3xy - 10y^2 = \underline{(x - 2y)(x + 5y)}$
   (e) $12x^2 - 29x + 15 = \underline{(3x - 5)(4x - 3)}$
   (f) $2a^2 + 7ab + 3b^2 = \underline{(2a + b)(a + 3b)}$

2. (a) $2x^2 + 11 + 12 = (2x + 3)\underline{(x + 4)}$
   (b) $6x^2 + 19x + 15 = (3x + 5)\underline{(2x + 3)}$
   (c) $8x^2 + 10x - 3 = (4x - 1)\underline{(2x + 3)}$
   (d) $10x^2 - 29x + 10 = (2x - 5)\underline{(5x - 2)}$
   (e) $15x^2 - 29x - 14 = (5x + 2)\underline{(3x - 7)}$
   (f) $8x^2 + 14x - 15 = (4x - 3)\underline{(2x + 5)}$

3. (a) $2x^2 + x - 6 = \underline{(2x - 3)(x + 2)}$
   (b) $12x^2 - 8x - 15 = \underline{(6x + 5)(2x - 3)}$
   (c) $12y^2 + 2y - 14 = 2(6y^2 + y - 7)$ $= \underline{2(6y + 7)(y - 1)}$
   (d) $4x^2 + 12x + 9 = \underline{(2x + 3)(2x + 3)}$
   (e) $4x^2 - 4x + 1 = \underline{(2x - 1)(2x - 1)}$
   (f) $9x^2 - 36x + 27 = 9(x^2 - 4x + 3)$ $= \underline{9(x - 3)(x - 1)}$
   (g) $-6x^2 + 5x + 4 = -1(6x^2 - 5x - 4)$ $= \underline{-(3x - 4)(2x + 1)}$
   (h) $6x^2 + 7xy - 5y^2 = \underline{(3x + 5y)(2x - y)}$
   (i) $4x^4 - 4x^2 - 35 = \underline{(2x^2 - 7)(2x^2 + 5)}$
   (j) $8y^2 + 8y - 30 = 2(4y^2 + 4y - 15)$ $= \underline{2(2y - 3)(2y + 5)}$

Exercise #7, page 33

1. (a) yes (b) no; fractional exponent (c) yes

   (d) yes (e) no; This is an equation. (f) no; square root of a variable

2. (a) $x = 0$ (b) $x = 2$ or $x = -1$ (c) $x = 0$ or $x = 3$

   (d) $\frac{5}{x(x + 3)}$ (Undefined when $x = 0$ or $x = -3$.)

   (e) $\frac{(x + 1)^2}{(x - 5)(x + 2)}$ (Undefined when $x = 5$ or $x = -2$.)

   (f) $\frac{x^2 + 3}{(x - 4)(x + 4)}$ (Undefined when $x = 4$ or $x = -4$.)

Exercise #8, page 38

1.

$\frac{5x^2 + 5x}{10x^3}$	$\frac{5x(x + 1)}{10x^3}$	5, x	$\frac{\cancel{5}\cancel{x}(x + 1)}{\underset{2}{\cancel{10}}\cancel{x^3}\,x^2} = \frac{x + 1}{2x^2}$
$\frac{x^2 + x - 20}{3x + 15}$	$\frac{(x + 5)(x - 4)}{3(x + 5)}$	(x + 5)	$\frac{\cancel{(x + 5)}(x - 4)}{3\cancel{(x + 5)}} = \frac{x - 4}{3}$
$\frac{6x^2 + 7x - 5}{3x^2 + 2x - 5}$	$\frac{(3x + 5)(2x - 1)}{(3x + 5)(x - 1)}$	(3x + 5)	$\frac{\cancel{(3x + 5)}(2x - 1)}{\cancel{(3x + 5)}(x - 1)} = \frac{2x - 1}{x - 1}$
$\frac{18a^2 + 21a - 60}{27a^2 - 48}$	$\frac{3(3a - 4)(2a + 5)}{3(3a - 4)(3a + 4)}$	3, (3a - 4	$\frac{\cancel{3}\cancel{(3a - 4)}(2a + 5)}{\cancel{3}\cancel{(3a - 4)}(3a + 4)} = \frac{2a + 5}{3a + 4}$
$\frac{t^2 - 7t}{t^2 - 8t + 7}$	$\frac{t(t - 7)}{(t - 7)(t - 1)}$	(t - 7)	$\frac{t\cancel{(t - 7)}}{\cancel{(t - 7)}(t - 1)} = \frac{t}{t - 1}$
$\frac{4ab^2 + 8b^3}{a^2 - ab - 6b^2}$	$\frac{4b^2(a + 2b)}{(a - 3b)(a + 2b)}$	(a + 2b)	$\frac{4b^2\cancel{(a + 2b)}}{(a - 3b)\cancel{(a + 2b)}} = \frac{4b^2}{a - 3b}$

Exercise #8, continued

2. (a) yes; a and -a are additive inverses.
   -b and b are additive inverses.

   (b) no; x and x are not additive inverses.

   (c) no; This expression equals 1.

   (d) yes; 3a and -3a are additive inverses.
   4b and -4b are additive inverses.

   (e) yes; -r and r are additive inverses.
   2s and -2s are additive inverses.

   (f) no; -5b and -5b are not additive inverses.

   (g) yes; -x and x are additive inverses.
   -y and y are additive inverses.

   (h) yes; y and -y are additive inverses.
   -2x and 2x are additive inverses.

3. (a) $\dfrac{-\overset{9}{\cancel{36}}\overset{a}{\cancel{a^2}}\overset{1}{\cancel{b^2}}}{-\underset{1}{\cancel{4}}\underset{1}{\cancel{a}}\underset{b^2}{\cancel{b^4}}c} = \boxed{\dfrac{9a}{b^2c}}$

   (b) $\dfrac{6x}{2x^3 - 2xy} = \dfrac{\overset{3}{\cancel{6x}}}{\underset{1}{\cancel{2x}}(x^2 - y)} = \boxed{\dfrac{3}{x^2 - y}}$

   (c) $\dfrac{3m - 3n}{2n - 2m} = \dfrac{3(\overset{-1}{\cancel{m - n}})}{2(\underset{1}{\cancel{n - m}})} = \boxed{\dfrac{-3}{2}}$

   (d) $\dfrac{6 - t}{t^2 - 36} = \dfrac{(\overset{-1}{\cancel{6 - t}})}{(\underset{1}{\cancel{t - 6}})(t + 6)} = \boxed{\dfrac{-1}{t + 6}}$

   (e) $\dfrac{5x^2 - 5y^2}{x^2 - 2xy + y^2} = \dfrac{5(x^2 - y^2)}{(x - y)^2}$

   $= \dfrac{5(\overset{1}{\cancel{x - y}})(x + y)}{(\underset{1}{\cancel{x - y}})(x - y)}$

   $= \boxed{\dfrac{5(x + y)}{(x - y)}}$

   (f) $\dfrac{4b - 16a}{16a^2 - b^2} = \dfrac{4(\overset{-1}{\cancel{b - 4a}})}{(\underset{1}{\cancel{4a - b}})(4a + b)}$

   $= \boxed{\dfrac{-4}{4a + b}}$

Exercise #9, page 42

1. (a) 2, x^2, y

   (b) 9, z^3, y

   (c) 3, y^2, z

   (d) (x + 4), (x - 4)

   (e) 2, (y + 5)

Exercise #9, continued

2.

$\frac{k^2 - k}{k^2 - 4} \times \frac{k^2 + 2k}{k^2 - k}$	$\frac{k(k - 1)\ k(k + 2)}{(k - 2)(k + 2)\ k(k - 1)}$	$k,\ (k - 1),\ (k + 2)$
$\frac{m^2 + 3m}{m^2 - 9} \times \frac{m^2 - 3m}{m^2}$	$\frac{m(m + 3)\ m(m - 3)}{(m + 3)(m - 3)m^2}$	$m^2,\ (m + 3),\ (m - 3)$
$\frac{4x^2 - 12x + 9}{2x - 3} \times \frac{3x - 5}{4x^2 - 9}$	$\frac{(2x - 3)(2x - 3)(3x - 5)}{(2x - 3)(2x - 3)(2x + 3)}$	$(2x - 3)^2$
$\frac{x^2 - x - 12}{x^2 - 2x - 3} \times \frac{x^2 + 5x + 4}{x^2 - 16}$	$\frac{(x - 4)(x + 3)(x + 1)(x + 4)}{(x - 3)(x + 1)(x - 4)(x + 4)}$	$(x - 4),\ (x + 1),\ (x + 4)$
$\frac{3m - 21}{6m + 12} \times \frac{2m^2 + 8m + 8}{m^2 - 49}$	$\frac{3(m - 7)\ 2(m + 2)(m + 2)}{6(m + 2)(m - 7)(m + 7)}$	$6,\ (m - 7),\ (m + 2)$

3. (a) $\frac{\overset{1}{\cancel{2}}\overset{1}{\cancel{x}} \times \overset{3y}{\cancel{9}\cancel{y^2}}}{\underset{1}{\cancel{3}}\underset{1}{\cancel{y}} \times \underset{4x^2}{\cancel{8}\cancel{x^3}}} = \boxed{\frac{3y}{4x^2}}$

(b) $\frac{2x \times -\overset{-1}{\cancel{5}}\overset{1}{\cancel{y}}}{\underset{1}{\cancel{5}}\underset{y}{\cancel{y^2}}} = \boxed{\frac{-2x}{y}}$

(c) $\frac{5\overset{-1}{\cancel{(z - y)}}\overset{1}{\cancel{(y + z)}}}{3\underset{1}{\cancel{(y + z)}}\underset{1}{\cancel{(y - z)}}} = \boxed{\frac{-5}{3}}$

(d) $\frac{(x - 1)\overset{1}{\cancel{(x + 1)}}\overset{1}{\cancel{x}}\overset{1}{\cancel{(x + 3)}}}{(x - 3)\underset{1}{\cancel{(x + 3)}}\underset{1}{\cancel{x}}\underset{1}{\cancel{(x + 1)}}} = \boxed{\frac{x - 1}{x - 3}}$

(e) $\frac{\overset{-1}{\cancel{(3k + 2t)}}\overset{1}{\cancel{t}}}{\underset{1}{\cancel{t}}\underset{1}{\cancel{(-3k - 2t)}}} = \boxed{-1}$

(f) $\frac{(c - 3)\overset{1}{\cancel{(c - 2)}}\overset{1}{\cancel{(c + 3)}}}{c\underset{1}{\cancel{(c + 3)}}\underset{1}{\cancel{(c - 2)}}} = \boxed{\frac{c - 3}{c}}$

Exercise #10, page 46

1. (a) $\frac{5a^2c^2}{b^2} \times \frac{3b^3}{-20ac} = \frac{\overset{1}{\cancel{5}}\overset{a}{\cancel{a^2}}\overset{c}{\cancel{c^2}} \times 3\overset{b}{\cancel{b^3}}}{\underset{1}{\cancel{b^2}} \times -\underset{-4}{\cancel{20}}\underset{1}{\cancel{a}}\underset{1}{\cancel{c}}} = \frac{3abc}{-4} = \boxed{\frac{-3abc}{4}}$

(b) $\frac{m^2 - n^2}{mn} \times \frac{1}{m - n} = \frac{(m + n)\overset{1}{\cancel{(m - n)}}}{mn\underset{1}{\cancel{(m - n)}}} = \boxed{\frac{m + n}{mn}}$

Exercise #10, continued

(c) $\frac{-xy^2}{1 - x} \times \frac{x - 1}{2y} = \frac{-x\overset{y}{\cancel{y^2}}(\overset{-1}{\cancel{x - 1}})}{(\underset{1}{\cancel{1 - x}})2\underset{1}{\cancel{y}}} = \boxed{\frac{xy}{2}}$

(d) $\frac{x^2 - xy}{x^2 + xy} \times \frac{x + y}{x^2 - y^2} = \frac{\overset{1}{\cancel{x}}(\overset{1}{\cancel{x - y}})(\overset{1}{\cancel{x + y}})}{\underset{1}{\cancel{x}}(\underset{1}{\cancel{x + y}})(x + y)(\underset{1}{\cancel{x - y}})} = \boxed{\frac{1}{x + y}}$

Exercise #11, page 53

1. (a) m^2n^2 (b) abc (c) $(x + y)^2(x - y)$

(d) $xy(x + y)$ (e) $12a^2b^2$ (f) $t(t - 1)^2(t + 1)^2$

2. (a) L.C.D. is $\underline{24}$.

$\frac{a}{4} = \frac{a \times 6}{4 \times 6} = \boxed{\frac{6a}{24}}$

$\frac{b}{6} = \frac{b \times 4}{6 \times 4} = \boxed{\frac{4b}{24}}$

$\frac{c}{8} = \frac{c \times 3}{8 \times 3} = \boxed{\frac{3c}{24}}$

(b) L.C.D. is $\underline{xyz}$.

$\frac{z}{xy} = \frac{z \times z}{xy \times z} = \boxed{\frac{z^2}{xyz}}$

$\frac{x}{yz} = \frac{x \times x}{yz \times x} = \frac{x^2}{xyz}$

$\frac{y}{zx} = \frac{y \times y}{zx \times y} = \boxed{\frac{y^2}{xyz}}$

(c) $x^2 - 7x + 12 = (x - 4)(x - 3)$
$x^2 - 9 = (x - 3)(x + 3)$
L.C.D. is $\underline{(x - 4)(x - 3)(x + 3)}$.

$$\frac{x + 2}{(x - 4)(x - 3)} = \frac{(x + 2)(x + 3)}{(x - 4)(x - 3)(x + 3)}$$

$$\frac{x + 4}{(x + 3)(x - 3)} = \frac{(x + 4)(x - 4)}{(x + 3)(x - 3)(x - 4)}$$

(d) $x^2 - 25 = (x - 5)(x + 5)$
$2x + 10 = 2(x + 5)$
L.C.D. is $\underline{2(x - 5)(x + 5)}$.

$$\frac{x}{(x + 5)(x - 5)} = \frac{2x}{2(x + 5)(x - 5)}$$

$$\frac{1}{2(x + 5)} = \frac{(x - 5)}{2(x + 5)(x - 5)}$$

3. (a) $\frac{3y - (2y - 7) + 5}{z} = \frac{3y + (-2y + 7) + 5}{z} = \boxed{\frac{y + 12}{z}}$

(b) $\frac{-3x + y}{x - y} + \frac{(-1)(2y - x)}{(-1)(y - x)} = \frac{-3x + y}{x - y} + \frac{-2y + x}{x - y} + \frac{-3x + y - 2y + x}{x - y} = \boxed{\frac{-2x - y}{x - y}}$

(c) $\frac{(a + b)4}{15 \times 4} + \frac{(a - b)3}{20 \times 3} + \frac{2b \times 15}{4 \times 15} = \frac{4a + 4b + 3a - 3b + 30b}{60} = \boxed{\frac{7a + 31b}{60}}$

Exercise #11, continued

(d) $\frac{3}{x^2} + \frac{5 \times x}{x \times x} - \frac{2 \times x^2}{1 \times x^2} = \boxed{\frac{3 + 5x - 2x^2}{x^2}}$

(e) $\frac{y^2 - 2}{(y - 2)(y + 1)} - \frac{y + 1}{y - 2} = \frac{y^2 - 2}{(y - 2)(y + 1)} - \frac{(y + 1)\boxed{(y + 1)}}{(y - 2)\boxed{(y + 1)}}$ ← MULTIPLY BOTH TERMS BY (y+1).

$$= \frac{(y^2 - 2) - (y + 1)(y + 1)}{(y - 2)(y + 1)}$$

$$= \frac{(y^2 - 2) - (y^2 + 2y + 1)}{(y - 2)(y + 1)}$$

$$= \frac{(y^2 - 2) + (-y^2 - 2y - 1)}{(y - 2)(y + 1)}$$

$$= \boxed{\frac{-2y - 3}{(y - 2)(y + 1)}}$$

(f) $\frac{2a}{(a - 3)(a + 3)} + \frac{1 \times (a + 3)}{(a - 3)(a + 3)} - \frac{1 \times (a - 3)}{(a + 3)(a - 3)}$

$$= \frac{2a + (a + 3) - (a - 3)}{(a + 3)(a - 3)}$$

$$= \frac{2a + a + 3 - a + 3}{(a + 3)(a - 3)}$$

$$= \frac{2a + 6}{(a + 3)(a - 3)}$$

$$= \frac{2\overset{1}{\cancel{(a + 3)}}}{\underset{1}{\cancel{(a + 3)}}(a - 3)}$$

$$= \boxed{\frac{2}{a - 3}}$$

End of Lesson E

Lesson F

EQUATIONS AND INEQUALITIES IN ONE VARIABLE

Topic One: Linear Equations in One Variable

A. Equations

An EQUATION is a statement of equality between numerical or variable expressions. All equations contain the relations symbol "=". The expression to the left of the equals sign is called the LEFT MEMBER of the equation while the expression to the right is called the RIGHT MEMBER.

EXAMPLES:

$$8 + 7 \overset{\text{EQUALS SIGN}}{=} 3 \times 5$$

$$4x + 5 = -13$$

$$\underset{\text{LEFT MEMBER}}{y} = \underbrace{x^2 - 2x + 3}_{\text{RIGHT MEMBER}}$$

Some equations involve real numbers only and are either true or false.

EXAMPLES:

1. $2 \times 6 = 36 \div 3$ is a number equation which is true.
   (Note that both sides equal 12.)

2. $2 + 2 = 3 + 3$ is a number equation which is false.
   (Note that the left side equals 4 and the right side equals 6.)

Other equations involve at least one variable term and become true or false only when specific values have been assigned to the variables involved. Such equations are classified as being either conditional equations or identities.

A CONDITIONAL EQUATION is true for some replacements of the variables involved and false for others.

EXAMPLES:

1. $x - 7 = 3$ is a conditional equation involving the one variable x. If we assign the value 4 to x, we obtain the number equation $4 - 7 = 3$ which is false. On the other hand, if we assign the value 10 to x, we obtain the number equation $10 - 7 = 3$ which is true.

2. $x + y = 8$ is a conditional equation involving the two variables x and y. If we assign the value 3 to x and the value 4 to y, we obtain the number equation $3 + 4 = 8$ which is false. On the other hand, if we assign the value 2 to x and the value 6 to y, we obtain the number equation $2 + 6 = 8$ which is true.

AN IDENTITY is an equation that is true for all replacements of the variables.
EXAMPLES:

1. $x + 5 = 5 + x$ is an identity involving the one variable x. This identity is true for all replacements from the domain of x.
   For example,
   - if $x = 2$, $2 + 5 = 5 + 2$ is a true statement since both sides equal 7.
   - if $x = -5$, $-5 + 5 = 5 + (-5)$ is a true statement since both sides equal 0.
   - if $x = 8$, $8 + 5 = 5 + 8$ is a true statement since both sides equal 13

   and so on.

2. $2(a + b) = 2a + 2b$ is an identity involving the two variables a and b. This identity is true for any real values that are substituted for a and b. For example,
   - if $a = 1$ and $b = 3$, $2(1 + 3) = (2 \times 1) + (2 \times 3)$ is a true statement since both sides equal 8.
   - if $a = -3$ and $b = 4$, $2(-3 + 4) = (2 \times -3) + (2 \times 4)$ is a true statement since both sides equal 2.

Self-correcting Exercise #1

Answers may be found on page 63 of this lesson.

1. State whether each of the following is a number equation, conditional equation, or an identity.

(a) $2 \times -3 = -3 \times 2$ ______
(b) $\frac{x}{3} = 7$ ______
(c) $7 = 6 + 2$ ______
(d) $(x - 5)(x + 1) = x^2 - 4x - 5$ ______
(e) $8 - y^2 = x$ ______
(f) $ab \times c = c \times ab$ ______

2. Decide whether each equation is a true or false statement for the values of the variables that are given.

True or False?

(a) $7x - 5 = 9$ when $x = 2$ ______
(b) $x^2 + 1 = 9$ when $x = -3$ ______
(c) $xy + y^2 = x^2$ when $x = 1$ and $y = 3$ ______
(d) $5x^2 + yx = 8$ when $x = -1$ and $y = -3$ ______
(e) $3mn - 2 = 7$ when $m = 2$ and $n = 3$ ______
(f) $x(y + 1) = xy + x$ when $x = 1$ and $y = 0$ ______

B. Classifying Conditional Equations

A conditional equation can be classified according to the number of different variables involved in the equation and the degree of the equation.

The degree of a conditional equation can be determined by noting the degree of its individual terms. Recall that the degree of a term is found by adding the exponents of all the variables which appear in the term. The degree of an equation is the same as its term with the highest degree. Special names are assigned to conditional equations of degree one, two, three, and four. An equation of degree one is called a LINEAR EQUATION, an equation of degree two is called a QUADRATIC EQUATION, an equation of degree three is called a CUBIC EQUATION, and an equation of degree four is called a QUARTIC EQUATION.

EXAMPLES:

1. $3x + 2 = 5x$
   This is a linear equation in one variable. (This equation involves only the one variable x. It is a linear equation because the terms $3x$ and $5x$ are of degree one.)

2. $2y^2 = 8 - x$
   This is a quadratic equation in two variables. (This equation involves the two variables x and y. It is a quadratic equation because the term $2y^2$ is of degree two.)

3. $x^4 + y^2 + z = 0$
   This is a quartic equation in three variables. (This equation involves the three variables x, y, and z. It is a quartic equation because the term x^4 is of degree four.)

4. $x^3 + 2x^2 - 5x + 7 = 0$
   This is a cubic equation in one variable. (This equation involves the one variable x. It is a cubic equation because the term x is of degree three.)

Self-correcting Exercise #2

Answers may be found on page 63 of this lesson.

1. Classify each equation according to its degree and the number of variables involved.

	Classification
(a) $x^2 = 9x$	quadratic in one variable
(b) $\frac{4}{5}x = \frac{2}{3}$	
(c) $mn = 3$	
(d) $a^3 = ab + 3$	
(e) $x^2 + xyz = z^2$	
(f) $y - 4 = y^2 + 2y$	
(g) $x^4 - 2x^2y^2 + y^4 = 0$	
(h) $a^2 + b^2 = c^2$	
(i) $x^3 = -x^2 + 3x$	
(j) $x^2 + y^2 = 25$	
(k) $5x - x^2 = 3$	
(l) $y = x - 5$	

C. The Solution Set of a Linear Equation in One Variable

A LINEAR EQUATION IN ONE VARIABLE is a conditional equation of degree one which involves only one variable. The following are examples of equations of this type.

$$x = 7$$

$$x - 5 = 2$$

$$2x + 3 = x - 4$$

$$\frac{x}{4} + 2 = \frac{x}{3}$$

In dealing with linear equations in one variable, we are interested in finding the one value from the domain of the variable which will make the equation a true statement. This value is called the ROOT or SOLUTION of the equation. The subset of the domain which consists of the one value of the variable that makes the equation true is called the SOLUTION SET of the equation.

EXAMPLE: If $x \in R$, find the solution set of the equation $\frac{x}{3} = -4$.

Solution

We are seeking a value of x which when divided by 3 will equal -4. The only possible value of x that satisfies this condition is -12.

i.e. when $x = -12$, $\frac{-12}{3} = -4$

$$-4 = -4$$

(true statement)

Since we obtain a true statement when $x = -12$, we know that -12 is a root of the equation $\frac{x}{3} = -4$. Thus, the solution set of the equation is $\{-12\}$.

i.e. $\left\{x \mid \frac{x}{3} = -4,\ x \in R\right\} = \{-12\}$

Self-correcting Exercise #3

Answers may be found on page 63 of this lesson.

1. Fill in the blanks to make each statement true.

(a) $x + 4 = 9$ is satisfied when $x =$ ____ since ____ $+ 4 = 9$ is a true statement.

(b) $x - 3 = 7$ is satisfied when $x =$ ____ since ____ $- 3 = 7$ is a true statement.

(c) $-3x = 21$ is satisfied when $x =$ ____ since $-3 \times$ ____ $= 21$ is a true statement.

(d) $15 - x = 6$ is satisfied when $x =$ ____ since $15 -$ ____ $= 6$ is a true statement.

(e) $\frac{x}{-2} = 8$ is satisfied when $x =$ ____ since $\frac{\quad}{-2} = 8$ is a true statement.

2. Solve each of the following equations by inspection. Then state the solution set.

(a) $x + 18 = 29$

$x =$ ______

Solution set is $\{11\}$.

(b) $x - 14 = 30$

$x =$ ______

Solution set is ______.

(c) $96 - x = 27$

$x =$ ______

Solution set is ______.

(d) $\frac{x}{5} = 90$

$x =$ ______

Solution set is ______.

(e) $\frac{63}{x} = 7$

$x =$ ______

Solution set is ______.

(f) $3x - 1 = 8$

$x =$ ______

Solution set is ______.

D. Formal Method of Solving Linear Equations in One Variable

The solution set of a linear equation in one variable cannot always be easily determined by inspection. For example, the root of the equation $4x - 7 = 6x + 5$ is not obvious at first glance. When linear equations become more involved, they must be solved by a more formal method. The solution set of any equation of the form $x = k$ where $k \in R$ is $\{k\}$. For example, the solution set of $x = 3$ is $\{3\}$, the solution set of $x = \sqrt{2}$ is $\{\sqrt{2}\}$, and so on. Thus, if we can put any linear equation in one variable in the form $x = k$, we can easily determine its solution set.

Consider the following equations.

$$2x + 7 = 15 \quad (1)$$
$$2x = 8 \quad (2)$$
$$x = 4 \quad (3)$$

By inspection, we can tell that the solution set of each of these equations is $\{4\}$. Note that equation (2) can be derived from equation (1) by subtracting 7 from each side of equation (1).

i.e. $2x + 7 = 15 \quad (1)$

Subtract 7 from each side.

$$2x + 7 - 7 = 15 - 7$$
$$2x = 8 \quad (2)$$

Equation (3) can be derived from equation (2) by dividing each side of equation (2) by 2.

i.e. $2x = 8 \quad (2)$

Divide each side by 2.

$$\frac{2x}{2} = \frac{8}{2}$$

$$x = 4 \quad (3)$$

This example suggests a formal method for solving all linear equations in one variable. Write a series of equations with the same solution set as the original until you arrive at an equation of the form $x = k$ whose solution set is $\{k\}$. Equations which have the same solution set are called EQUIVALENT EQUATIONS.

In our example, we were able to obtain equivalent equations by subtracting 7 from each side of the original equation and then dividing each side of the equation by 2. In general, we will obtain an equivalent equation if we perform the same operation of addition, subtraction, multiplication, or division on each side of a given equation.

PROPERTIES OF EQUALITY

1. Addition Property
   $a = b$ and $a + c = b + c$ are equivalent equations.
   (This means that if the same real number is added to each side of an equation, an equivalent equation is obtained.)
2. Subtraction Property
   $a = b$ and $a - c = b - c$ are equivalent equations.
   (This means that if the same real number is subtracted from each side of an equation, an equivalent equation is obtained.)
3. Multiplication Property
   $a = b$ and $ac = bc$ (where $c \neq 0$) are equivalent equations.
   (This means that if each side of an equation is multiplied by the same non-zero real number, an equivalent equation is obtained.)
4. Division Property
   $a = b$ and $\frac{a}{c} = \frac{b}{c}$ (where $c \neq 0$) are equivalent equations.
   (This means that if each side of an equation is divided by the same non-zero real number, an equivalent equation is obtained.)

The solution set of a linear equation in one variable can be determined by writing a series of equivalent equations until you arrive at an equation of the form $x = k$ where $k \in R$. You can use the properties of equality to help you write this series of equivalent equations, but you must have some way of knowing which property to use and what number to operate with.

You can use the idea of inverse operations to help you decide which operation to use in solving an equation. Remember that addition and subtraction are inverse operations and multiplication and division are inverse operations. In order to arrive at an equation of the form $x = k$, you must undo any operation in the given equation which links a real number to x.

1. Use the operation of subtraction to undo an addition.

EXAMPLE: $x + 2 = 6$

In order to undo the operation of adding 2 to x, we must subtract 2 from both sides of the equation.

SUBTRACT 2.

$$x + 2 - 2 = 6 - 2$$

$$x = 4$$

The solution set is $\{4\}$.

2. Use the operation of addition to undo a subtraction.

EXAMPLE: $x - 2 = -4$

In order to undo the operation of subtracting 2 from x, we must add 2 to both sides of the equation.

ADD 2.

$$x - 2 + 2 = -4 + 2$$

$$x = -2$$

The solution set is $\{-2\}$.

3. Use the operation of division to undo a multiplication.

EXAMPLE: $\frac{2}{3}x = 12$

In order to undo the operation of multiplying x by $\frac{2}{3}$, divide each side of the equation by $\frac{2}{3}$.

$$\frac{\frac{\cancel{2}^{1}}{\cancel{3}}x}{\frac{\cancel{2}}{\cancel{3}_{1}}} = \frac{12}{\frac{2}{3}}$$

DIVIDING BY 2/3 IS EQUIVALENT TO MULTIPLYING BY 3/2.

$$x = 12 \times \left(\frac{3}{2}\right)$$

$$x = 18$$

The solution set is $\{18\}$.

4. Use the operation of multiplication to undo a division.

EXAMPLE: $\frac{x}{3} = \frac{5}{8}$

In order to undo the operation of dividing x by 3, we must multiply both sides of the equation by 3.

$$\cancel{3}^{1} \times \frac{x}{\cancel{3}_{1}} = 3 \times \frac{5}{8}$$

$$x = \frac{15}{8}$$

The solution set is $\left\{\frac{15}{8}\right\}$.

Self-correcting Exercise #4

Answers may be found on page 64 of this lesson.

1. In each question describe the operation that was used to obtain equation (2) from equation (1).

(a) $2x = 9$ (1)
$x = \frac{9}{2}$ (2) Divide each side by 2. ______

(b) $3 + x = -4$ (1)
$x = -7$ (2) ______

(c) $-2 + x = -8$ (1)
$x = -6$ (2) ______

(d) $\frac{x}{-2} = \frac{3}{4}$ (1)
$x = \frac{-3}{2}$ (2) ______

(e) $\frac{3}{4}x = -9$ (1)
$x = -12$ (2) ______

2. Use the properties of equality to find the solution set of each equation.

(a) $x - 7 = 3$
$x = 3+7$
$x = 10$

Solution set is $\{10\}$.

(b) $\frac{x}{4} = -1$

Solution set is ______.

(c) $x + 9 = 34$

Solution set is ______.

(d) $-5x = 35$

Solution set is ______.

(e) $\frac{3}{5}x = \frac{9}{25}$

Solution set is ______.

(f) $\frac{x}{5} = 0$

Solution set is ______.

E. Solutions Which Involve Many Operations

In some equations, the variable is operated on in more than one way. You will obtain the same result regardless of the order in which you undo the operations, but additions and subtractions are generally undone before multiplications or divisions.

EXAMPLE: Find the solution set of the equation $3 - 9x = -6$.
Solution

$$3 - 9x = -6$$
Subtract 3 from each side.
$$-9x = -9$$
Divide each side by -9.
$$x = 1$$
The solution set is $\{1\}$.

In some equations, you will find that a variable term appears on both sides of the equation. When solving such equations, you must add or subtract the appropriate variable term so that the variable occurs only on the left side of the equation.

EXAMPLE: Find the solution set of the equation $5x = 7x - 12$.
Solution

$$5x = 7x - 12$$
In order to remove the variable term 7x from the right side, you must subtract 7x from each side.
$$5x - 7x = 7x - 12 - 7x$$
$$-2x = -12$$
Now, divide each side by -2.
$$x = 6$$
The solution set is $\{6\}$.

Now, solve the following equations.

1. $5x = 2x + 3$
Subtract 2x from both sides.

$\underline{\qquad} = 3$

Divide both sides by 3.

$x = \underline{\qquad}$

2. $-2x + 12 = 4x$
Subtract 4x from both sides.

$\underline{\qquad} + 12 = 0$

Subtract 12 from both sides.

$\underline{\qquad} = -12$

Divide both sides by -6.

$x = \underline{\qquad}$

3. $2x = 3x + 7$

Subtract 3x from both sides.

$-x =$ ____

Multiply both sides by -1.

$x =$ ____

4. $-2x + 8 = 16 - 6x$

Add 6x to both sides.

____ $+ 8 = 16$

Subtract 8 from both sides.

____$x =$ ____

Divide both sides by 4.

$x =$ ____

Some linear equations contain a number of first degree terms and a number of constant terms on each side of the equation. When solving such equations you must first combine any like terms which appear on one side of the equation. Then, apply the properties of equality so that the left side contains only variable terms and the right side contains only constant terms.

EXAMPLE: Find the solution set of the equation $2(x - 5) + 3x = -3(x + 7) - 4$.

Solution

First, simplify each side of the equation and combine like terms.

$$\begin{aligned} 2(x - 5) + 3x &= -3(x + 7) - 4 \\ 2x - 10 + 3x &= -3x - 21 - 4 \\ 5x - 10 &= -3x - 25 \end{aligned}$$

Now, add 3x to each side in order to eliminate the variable term from the right side.

$$8x - 10 = -25$$

Next, add 10 to each side in order to eliminate the constant term from the left side.

$$8x = -15$$

Finally, divide each side by 8.

$$x = \frac{-15}{8}$$

Solution set is $\left\{\frac{-15}{8}\right\}$.

Solve the following equations.

1. $3(x + 1) - 2 = -2(2x + 3) + 5$

Apply the distributive property.

___ + ___ - 2 = -4x - ___ + 5

Collect like terms.

___ + 1 = -4x - ___

Add 4x to both sides.

___ + 1 = -1

Subtract 1 from both sides.

___x = ___

Divide both sides by 7.

x = ___

2. $6x - 4 + 3x + 8 = 2x - 8 + 3x$

Collect like terms.

___ + 4 = 5x - ___

Subtract 5x from both sides.

___ + 4 = -8

Subtract 4 from both sides.

___x = ___

Divide both sides by 4.

x = ___

Some linear equations contain terms which are fractional in nature. When solving such equations, it is often convenient to clear the equation of fractions in the first step. This can be accomplished by multiplying each side of the equation by the L.C.D. of all the fractions involved.

EXAMPLE: Find the solution set of the equation $\frac{3x - 1}{7} - \frac{2x - 4}{5} = \frac{x+1}{10}$.

Solution

Note that the L.C.D. of all the fractions is 70.
Multiply both sides of the equation by 70.

$$70\left(\frac{3x - 1}{7} - \frac{2x - 4}{5}\right) = 70\left(\frac{x + 1}{10}\right)$$

Apply the distributive property to the left member.

$$\overset{10}{\cancel{70}}\left(\frac{3x - 1}{\underset{1}{\cancel{7}}}\right) - \overset{14}{\cancel{70}}\left(\frac{2x - 4}{\underset{1}{\cancel{5}}}\right) = \overset{7}{\cancel{70}}\left(\frac{x + 1}{\underset{1}{\cancel{10}}}\right)$$

(Note that in effect you have multiplied every term in the equation by the L.C.D., 70.)

$$\begin{aligned}
10(3x - 1) - 14(2x - 4) &= 7(x + 1)\\
30x - 10 - 28x + 56 &= 7x + 7\\
2x + 46 &= 7x + 7\\
-5x + 46 &= 7\\
-5x &= -39\\
x &= \frac{39}{5}
\end{aligned}$$

Solution set is $\left\{\frac{39}{5}\right\}$.

Solve the following equations.

1. $\frac{m}{5} = 6 + \frac{m}{8}$

Multiply every term by 40.

$\left(\overset{8}{\cancel{40}} \times \frac{m}{\underset{1}{\cancel{5}}}\right) = (40 \times 6) + \left(\overset{5}{\cancel{40}} \times \frac{m}{\underset{1}{\cancel{8}}}\right)$

$8m = \_\_\_\_ + 5m$

Subtract 5m from both sides.

$\_\_\_m = \_\_\_\_$

Divide both sides by 3.

$m = \_\_\_\_$

2. $\frac{2w + 5}{4} - \frac{10w + 3}{8} = 2w + 1$

Multiply every term by 8.

$\overset{2}{\cancel{8}}\left(\frac{2w + 5}{\underset{1}{\cancel{4}}}\right) - \overset{1}{\cancel{8}}\left(\frac{10w + 3}{\underset{1}{\cancel{8}}}\right) = 8(2w+1)$

$2(2w + 5) - (10w + 3) = 8(2w+1)$

Apply distributive property.

$4w + \_\_\_ - 10w - \_\_\_ = 16w + 8$

Collect like terms on left side.

$\_\_\_ + \_\_\_ = 16w + 8$

Subtract 16w from both sides.

$\_\_\_ + \_\_\_ = 8$

Subtract 7 from both sides.

$\_\_\_w = \_\_\_$

Divide both sides by -22.

$w = \_\_\_$

Self-correcting Exercise #5

Answers may be found on page 64 of this lesson.

1. Describe the operation that was performed at each step in the solutions.

(a) $3x + 7 = 4$

Subtract 7 from both sides.

$3x = -3$

____

$x = -1$

(b) $2x - 6 = 3x + 4$

____

$-x - 6 = 4$

____

$-x = 10$

____

$x = -10$

(c) $\frac{1}{2}x - \frac{2}{3} = \frac{x}{6}$

$3x - 4 = x$

$2x - 4 = 0$

$2x = 4$

$x = 2$

(d) $x + 3 - 5x = 8 - x - 9$

$-4x + 3 = -x - 1$

$-3x + 3 = -1$

$-3x = -4$

$x = \frac{4}{3}$

(e) $4(x + 3) - 4 = 2(x - 3)$

$4x + 12 - 4 = 2x - 6$

$4x + 8 = 2x - 6$

$2x + 8 = -6$

$2x = -14$

$x = -7$

(f) $\frac{x - 4}{4} = \frac{3x + 1}{6} - 2x$

$3(x - 4) = 2(3x + 1) - 24x$

$3x - 12 = 6x + 2 - 24x$

$3x - 12 = -18x + 2$

$21x - 12 = 2$

$21x = 14$

$x = \frac{2}{3}$

2. Find the solution set of each of the following equations.

(a) $2x - 8 = 7 + 3x$

(b) $\frac{x}{5} + \frac{2}{15} = \frac{1}{10}$

Solution set is ______.

Solution set is ______.

(c) $4(x - 2) = 3(2x - 6)$

(d) $\frac{55y + 7}{6} = 7 - \frac{23y - 1}{24}$

Solution set is ______.

Solution set is ______.

F. Checking the Solution to an Equation

Since it is possible that you may make a calculation error when solving an equation, you should check your answer by substituting it in both sides of the original equation. If the left and right members of the equation are indeed equal when the value is substituted, your answer should be correct.

EXAMPLE: Solve the equation $\frac{x + 3}{4} = \frac{x - 2}{8}$ and check your solution.

Solution

$$\frac{x + 3}{4} = \frac{x - 2}{8}$$

Multiply both sides by 8.

$$\overset{2}{\cancel{8}}\left(\frac{x + 3}{\underset{1}{\cancel{4}}}\right) = \overset{1}{\cancel{8}}\left(\frac{x - 2}{\underset{1}{\cancel{8}}}\right)$$

$$2(x + 3) = x - 2$$

$$2x + 6 = x - 2$$

$$x + 6 = -2$$

$$\boxed{x = -8}$$

The solution set is {-8}.

Check: If x = -8,

$\frac{x + 3}{4}$	$\frac{x - 2}{8}$
$\frac{-8 + 3}{4}$	$\frac{-8 - 2}{8}$
$\frac{-5}{4}$	$\frac{-10}{8}$
	$\frac{-5}{4}$

L.S. = R.S.

EXERCISE - Linear Equations in One Variable

1. Fill in the blanks.

(a) Equations which have the same solution set are called

__________________________ equations.

(b) The equation $a + 3 = 3 + a$ is called an ________________ because it is true for all values in the domain of a.

(c) The one root of the equation $x - 2 = -5$ is __________.

The solution set of this equation is __________.

(d) We can write the equation $-2x = 5$ in the form $x = k$ where $k \in R$ by

__________________ each side of the equation by __________.

(e) The equation $3x^2 + 5 = 7x$ is a _____________________ equation

because it is of degree __________.

(f) A fractional equation can be solved by multiplying every term by the

______________________________ of all the fractions involved.

2. Find the solution set of each equation. Then check your work by substituting each root in the original equation.

EXAMPLE:

$$2x - 5 = 7$$

$$2x = 12$$

$$x = 6$$

The solution set is $\{6\}$.

Check

$2x - 5$	7
$2(6) - 5$	
$12 - 5$	
7	

(a) $3y + 82 = 58 - y$

Check

$3y + 82$	$58 - y$
3(__) + 82	58 - ____
__ + 82	____
_____	

(b) $2x - 13 = 3x + 7$

Check

$2x - 13$	$3x + 7$
$2(\ \ ) - 13$	$3(\_\_) + 7$
___ $- 13$	___ $+ 7$
______	______

(c) $2x - 15 = 5x$

Check

$2x - 15$	$5x$

(d) $\frac{x}{4} + 7 = -3$

Check

$\frac{x}{4} + 7$	-3

(e) $3x - 2 + 4x + 7 = 8x - 6$

Check

$3x - 2 + 4x + 7$	$8x - 6$

(f) $3x - (4 + 5x) = 5(2x + 1)$

Check

$3x - (4 + 5x)$	$5(2x + 1)$

(g) $\frac{x + 2}{2} = \frac{x - 1}{3}$

Check

$\frac{x + 2}{2}$	$\frac{x - 1}{3}$

(h) $\frac{2x - 3}{5} - 7 = \frac{5x + 1}{10}$

Check

$\frac{2x - 3}{5} - 7$	$\frac{5x + 1}{10}$

(i) $\frac{x}{3} - \frac{x+5}{4} = \frac{2x+3}{6}$

Check

$\frac{x}{3} - \frac{x+5}{4}$	$\frac{2x+3}{6}$

3. Match each equation in Column II with its proper classification in Column I. (Classifications may be used more than once.)

Column I

A. Linear equation in one variable

B. Linear equation in two variables

C. Linear equation in three variables

D. Quadratic equation in one variable

E. Quadratic equation in two variables

F. Quadratic equation in three variables

G. Cubic equation in one variable

H. Cubic equation in two variables

I. Cubic equation in three variables

Column II

E 1. $y = 3x^2 - 5x$

___ 2. $x^3 - x^2 = 0$

___ 3. $xy = 7$

___ 4. $x - 7y = 3$

___ 5. $-3x + 5 = -7$

___ 6. $y = x^2z + 1$

___ 7. $x = 2$

___ 8. $9x^2 = 18$

___ 9. $x^2 + y^2 = 27$

___ 10. $a + b + c = 7$

___ 11. $5x^2 + 1 = 10$

___ 12. $y = x^3 - 7x + 3$

___ 13. $x^2 + y^2 = z^2$

___ 14. $x^2 - 2xy + y^2 = 0$

___ 15. $\frac{x}{3} = \frac{y}{5}$

Topic Two: Linear Inequalities in One Variable

A. Inequalities

Recall the meanings of the following symbols of inequality.

$>$ means "is greater than."
$\geqq$ means "is greater than or equal to."
$<$ means "is less than."
$\leqq$ means "is less than or equal to."

An INEQUALITY is a mathematical statement which uses one of the above symbols to link two numerical or variable expressions.

EXAMPLES:

$$6 + 2 < 9$$
$$3y - 7 \geqq 2$$
$$y \leqq x^2 - x$$

Like equations, some inequalities involve real numbers only and are either true or false.

EXAMPLES:

1. $2 + 3 < 8$ is a number inequality which is true.
2. $4 + 9 > 15$ is a number inequality which is false.

Other inequalities involve some variable terms and become true or false only when specific values have been assigned to the variables involved. Such inequalities are classified either as being conditional or unconditional.

A CONDITIONAL INEQUALITY is true for some replacements of the variables involved and false for others.

EXAMPLES:

1. $x - 2 < 5$ is a conditional inequality involving the one variable x. If we assign the value 4 to x, we obtain the number inequality $4 - 2 < 5$ which is true. On the other hand, if we assign the value 7 to x, we obtain the number inequality $7 - 2 < 5$ which is false. Decide whether the inequality $x - 2 < 5$ is true or false when:

(a) $x = 0$ true (b) $x = 8$ ________ (c) $x = -2$ ________

(d) $x = 6$ ________ (e) $x = 1$ ________ (f) $x = 9$ ________

2. $x + y > 6$ is a conditional inequality involving the two variables x and y. If we assign the value 0 to x and the value 7 to y, we obtain the number inequality $0 + 7 > 6$ which is true. On the other hand, if we assign the value 2 to x and the value 3 to y, we obtain the number inequality $2 + 3 > 6$ which is false. Decide whether the inequality $x + y > 6$ is true or false when:

   (a) $x = 1$ and $y = 3$? ________ (b) $x = 2$ and $y = 5$? ________

   (c) $x = -2$ and $y = -7$? ________ (d) $x = 15$ and $y = -4$? ________

An UNCONDITIONAL INEQUALITY is true for all replacements of the variables involved.

EXAMPLE:

$x^2 \geqq 0$ is an unconditional inequality that is true for all real values of x.
That is,

- if $x = \frac{1}{2}$, $\left(\frac{1}{2}\right)^2 \geqq 0$ is true since $\frac{1}{4} \geqq 0$ is true.
- if $x = -2$, $(-2)^2 \geqq 0$ is true since $4 \geqq 0$ is true.
- if $x = 3$, $3^2 \geqq 0$ is true since $9 \geqq 0$ is true.
- if $x = 0$, $0^2 \geqq 0$ is true since $0 \geqq 0$ is true.

Self-correcting Exercise #6

Answers may be found on page 65 of this lesson.

1. State whether each of the following is a number inequality, conditional inequality, or unconditional inequality.

   (a) $x + 3 < 5$ ________

   (b) $-9 + 2 > -3$ number

   (c) $|x| + 2 \geqq 2$ ________

   (d) $x^2 < 1$ ________

   (e) $-3 > -1 \times 5$ ________

   (f) $x^2 + 1 > 0$ ________

2. Decide whether each inequality is a true or false statement for the values of the variables that are given.

	True or False?
(a) $5m < 12$ when $m = 2$	
(b) $x^2 + 1 < 3$ when $x = 2$	
(c) $x < y + 1$ when $x = -5$ and $y = -4$	
(d) $x^2 + y^2 > 2$ when $x = 1$ and $y = 1$	
(e) $x^2 + y^2 > 25$ when $x = 4$ and $y = 10$	
(f) $x^2 \geqq x$ when $x = \frac{1}{2}$	

B. Classifying Conditional Inequalities

Conditional inequalities can be classified in the same manner as conditional equations.

EXAMPLES:

1. $5 - 2x \leqq 3x + 7$
   This is a linear inequality in one variable.
   (This inequality involves only the one variable x. It is linear because the terms -2x and 3x are of degree one.)

2. $y \geqq x^2 - 5x + 4$
   This is a quadratic inequality in two variables.
   (This inequality involves the two variables x and y. It is quadratic because the term x^2 is of degree two.)

Self-correcting Exercise #7

Answers may be found on page 65 of this lesson.

1. Classify each of the following inequalities according to its degree and the number of variables involved.

	Classification
(a) $3x^2 - 7x > 5$	quadratic in one variable
(b) $y \leqq x^3$	
(c) $2x + 7 < -3$	
(d) $x^2 + y^2 < 25$	

(e) $x^2 \geqq 1$

(f) $x + y < 5$

(g) $y \leqq 5 - x^2$

(h) $x^3 - 1 \geqq 0$

C. The Solution Set of a Linear Inequality in One Variable

A LINEAR INEQUALITY IN ONE VARIABLE is a conditional inequality of degree one which involves only one variable. The following are examples of inequalities of this type:

$$x < 5$$

$$x + 1 \geqq 5$$

$$2x + 1 \leqq 8x - 3$$

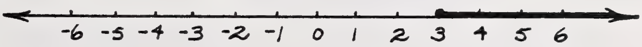
$$\frac{y}{4} > 9$$

In dealing with linear inequalities in one variable, we are interested in finding any values from the domain of the variable which will make the inequality a true statement. This subset of the domain which contains values of the variable that make the inequality true is called the SOLUTION SET of the inequality. Each individual member of the solution set is called a ROOT or SOLUTION of the inequality. The solution set of an inequality can be graphed on the number line.

EXAMPLE: If $x \in R$, find the solution set of the inequality $x - 2 \geqq 1$.

Solution

We are seeking a value of x which when decreased by 2 will be greater than or equal to 1. We know that if $x = 3$, the left member of the inequality equals 1. Thus, if we substitute any real number for x that is greater than or equal to 3, the left member will be greater than or equal to 1 and the inequality will be a true statement. Thus, the solution set of this inequality is $\{x | x \geqq 3, x \in R\}$. This solution set can be graphed on the number line as follows:



Self-correcting Exercise #8

Answers may be found on page 66 of this lesson.

1. Fill in the blanks.

   (a) The inequality $3x \leqq 15$ is satisfied when $x \leqq$ ______ since it becomes a true statement when any real number ____________ than or equal to ______ is substituted for x.
   The solution set is $\{x \mid$ ______, $x \in R\}$.

   (b) The inequality $x + 2 > 8$ is satisfied when $x >$ ______ since it becomes a true statement when any real number ____________ than ______ is substituted for x. The solution set is $\{x \mid$ ______, $x \in R\}$.

2. Write the solution set of each inequality in set-builder notation and then graph the set.

   (a) $5x > 10, \; x \in R$
   Solution Set:

   Graph:

   (b) $x + 5 \leqq 8, \; x \in R$
   Solution Set:

   Graph:

   (c) $\frac{x}{3} < 2, \; x \in R$
   Solution Set:

   Graph:

D. Formal Method for Solving Linear Inequalities in One Variable

The solution set of a linear inequality in one variable cannot always be easily determined by inspection. We will now devise a formal method for solving inequalities of this type. Our aim will be to write a series of equivalent inequalities (i.e. inequalities with the same solution set) until we arrive at one in which the variable appears alone in the left member and a real number appears alone in the right member. Before we can write this series of equivalent inequalities, we must know what operations can be performed on each side of an inequality in order to produce an equivalent inequality.

1. An equivalent inequality is obtained if we add the same real number to each side of a given inequality.

EXAMPLE: $x - 3 < 4$ (1)

Add 3 to each side.

$x < 7$ (2)

Note that inequalities (1) and (2) are equivalent since they both become true statements whenever any real number less than 7 is substituted for x.

2. An equivalent inequality is obtained if we subtract the same real number from each side of a given inequality.

EXAMPLE: $x + 2 \leqq 5$ (1)

Subtract 2 from each side.

$x \leqq 3$ (2)

Note that inequalities (1) and (2) are equivalent since they both become true statements whenever any real number that is less than or equal to 3 is substituted for x.

3. An equivalent inequality is not always obtained if we multiply each side of a given inequality by the same non-zero real number. An equivalent inequality is obtained if the multiplier is a positive real number but is not obtained if the multiplier is a negative real number.

EXAMPLE 1: $\frac{x}{2} < 5$ (1)

Multiply each side by 2.

$x < 10$ (2)

Note that inequalities (1) and (2) are equivalent since they both become true statements whenever any real number less than 10 is substituted for x.

EXAMPLE 2: $\frac{x}{-2} < 5$ (1)

Multiply each side by -2.

$x < -10$ (2)

Note that inequalities (1) and (2) are not equivalent. Inequality (1) becomes a true statement whenever any real number greater than -10 is substituted for x, while inequality (2) becomes a true statement whenever any real number less than -10 is substituted for x.

In example 2 above, we could have made the two inequalities equivalent if we had changed the direction of the arrow when we multiplied each side by the negative number -2.

i.e. $\frac{x}{-2} < 5$ (1)

Multiply each side by -2 and change the direction of the arrow.

$x > -10$ (2)

Now, these two inequalities are equivalent since they both become true statements when x is replaced by any real number greater than -10.

An equivalent inequality is obtained if we multiply each side of an inequality by the same positive real number, or if we multiply each side of an inequality by the same negative real number and change the direction of the arrow.

4. An equivalent inequality is not always obtained if we divide each side of a given inequality by the same non-zero real number. An equivalent inequality is obtained if the divisor is a positive real number but is not obtained if the divisor is a negative real number.

EXAMPLE 1: $2x > 6$ (1)

Divide each side by 2.

$x > 3$ (2)

Note that inequalities (1) and (2) are equivalent since they both become true statements whenever any real number greater than 3 is substituted for x.

EXAMPLE 2: $-2x > 6$ (1)

Divide each side by -2.

$x > -3$ (2)

Note that inequalities (1) and (2) are not equivalent. Inequality (1) becomes a true statement whenever any real number less than -3 is substituted for x, while inequality (2) becomes a true statement whenever any real number greater than -3 is substituted for x.

In example 2 above, we could have made the two inequalities equivalent if we had changed the direction of the arrow when we divided each side by the negative number -2.

i.e. $-2x > 6$ (1)

Divide each side by -2 and change the direction of the arrow.

$x < -3$ (2)

Now, these two inequalities are equivalent since they both become true statements when x is replaced by any real number less than -3.

An equivalent inequality is obtained if we divide each side of an inequality by the same positive real number, or if we divide each side of an inequality by the same negative real number and change the direction of the arrow.

Thus, linear inequalities can be solved in a similar manner to linear equations except for one major difference:

IF AN INEQUALITY IS MULTIPLIED OR DIVIDED BY THE SAME NEGATIVE REAL NUMBER, THE DIRECTION OF THE ARROW MUST BE CHANGED.

EXAMPLE: Find and graph the solution set of the inequality $5(3 - x) > 3(9 + x),\ x \in R$.

Solution

First, write each member of the inequality in simplest form.

$$15 - 5x > 27 + 3x$$

Subtract 3x from both sides.

$$15 - 8x > 27$$

Subtract 15 from both sides.

$$-8x > 12$$

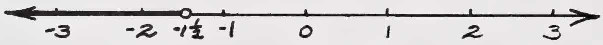
Divide each side by -8 and change direction of arrow.

$$x < \frac{-3}{2}$$

The solution set is $\{x \mid x < \frac{-3}{2},\ x \in R\}$.

Note: Since the domain of x in most inequalities is R, you need not write $x \in R$ as part of the solution set. This solution set could be written more simply as $\{x \mid x < \frac{-3}{2}\}$. You need only include the domain of the variable if it is anything other than R.

The solution set of $5(3 - x) > 3(9 + x)$ can be graphed on the real number line as follows:



$\{x \mid x < {}^{-}3/2\}$

NOTE: A hollow dot is placed at $-1\frac{1}{2}$ to indicate that $-1\frac{1}{2}$ is not included in the set. (We would use a solid dot if it were included.) The solid line to the left of $-1\frac{1}{2}$ represents all the real numbers less than $-1\frac{1}{2}$. The arrowhead indicates that the graph continues indefinitely to the left.

Self-correcting Exercise #9

Answers may be found on page 66 of this lesson.

1. In each question, describe the operation that was used to derive the second inequality from the first.

(a) $x - 4 > 9$ (1)
$x > 13$ (2)

Add 4 to both sides.

(b) $-y > 3$ (1)
$y < -3$ (2)

(c) $4 + x \leqq -2$ (1)
$x \leqq -6$ (2)

____

____

(d) $\frac{3}{5}x > 3$ (1)
$x > 5$ (2)

____

____

(e) $-9x \leqq 18$ (1)
$x \geqq -2$ (2)

____

____

(f) $x - 3 > -4$ (1)
$x > -1$ (2)

____

____

2. Describe the operation that was performed at each step in the solutions below.

(a) $15 > 4x + 12$

____

$-4x + 15 > 12$

____

$-4x > -3$

____

$x < \frac{3}{4}$

(b) $7(2x + 1) \leqq 5(3x - 4)$

____

$14x + 7 \leqq 15x - 20$

____

$-x + 7 \leqq -20$

____

$-x \leqq -27$

____

$x \geqq 27$

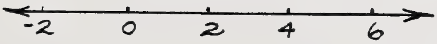
3. Use the properties of inequality to find the solution set of each inequality. (The domain of x is R.) Then graph the solution set.

(a) $15 > 2x + 7$

Solution set is

____

Graph:

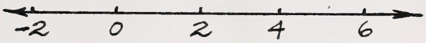


(b) $-3x + 6 \leqq 5x - 10$

Solution set is

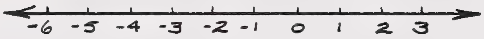
____

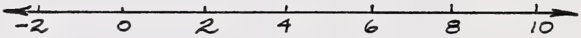
Graph:



(c) $2(1 - x) > -3(1 + x)$ (d) $\frac{x}{4} - \frac{x - 5}{6} \leqq \frac{x - 2}{3}$

Solution set is ________________. Solution set is ________________.





EXERCISE - Linear Inequalities in One Variable

1. Fill in the blanks.

(a) In order to solve the inequality $\frac{x}{-3} > 7$, we must ________________ both sides by ______ and change the ____________ of the arrow.

(b) $x + 3 < 9$ and $x < 6$ are ____________________ inequalities because they have the same ______________ set.

(c) If $x \in R$, the inequality $x - 3 < 7$ is true for all real values of x that are ________________ than ______.

(d) If we ________________ or ______________ both sides of an inequality by a ______________ real number, we must change the direction of the arrow.

(e) The solution set of the inequality $-2x \geqq 8$ can be written in set-builder notation as ___________________________.

2. Describe the operation that was performed at each step of the following solutions.

(a) $2 - 3x > -7$

____

$-3x > -9$

Divide each side by -3.
(Change arrow)

$x < 3$

(b) $\frac{1}{4}x - 5 < 2$

____

$\frac{1}{4}x < 7$

____

$x < 28$

(c) $-x - 3 > 4$

____

$-x > 7$

____

$x < -7$

(d) $-5x + 3 > 2x - 11$

____

$-7x + 3 > -11$

____

$-7x > -14$

____

$x < 2$

(e) $\frac{-3}{2}x + 7 \geqq 4$

____

$\frac{-3}{2}x \geqq -3$

____

$x \leqq 2$

(f) $\frac{2}{3}x \leqq \frac{1}{4}x - \frac{5}{6}$

____

$8x \leqq 3x - 10$

____

$5x \leqq -10$

____

$x \leqq -2$

3. Find and graph the solution set of each of the following inequalities. In each case, the domain of x is R.

(a) $-6x + 1 \leqq -5x - 2$

Solution set is ______________

Graph:

(b) $3x + 1 > 5x - 7$

Solution set is ______________.

Graph:

(c) $6x < 2(2x - 5) + 6$

Solution set is ______________.

Graph:

(d) $3x - 3(2x - 5) < 2x + 5$

Solution set is ______________.

Graph:

(e) $\frac{x}{2} + 2 < \frac{x}{4} + 1$

Solution set is ______________.

Graph:

(f) $\frac{x + 1}{2} + 2 \geqq \frac{5x - 3}{3}$

Solution set is ______________.

Graph:

Topic Three: Quadratic Equations in One Variable

A QUADRATIC EQUATION IN ONE VARIABLE is a conditional equation of degree two which involves only one variable. The following are examples of equations of this type.

$$\begin{aligned} x^2 &= 5 \\ x^2 + 3 &= 7x \\ 2x^2 &= 5x \\ 3x^2 - 7x &= 9 \\ x^2 - 7x &= 5x^2 + 9x - 2 \end{aligned}$$

A. Solving Quadratic Equations Which Can Be Put in the Form $ax^2 = c$ where $a, c \in R$

When solving linear equations in one variable, you learned that an equivalent equation is obtained whenever

(1) the same real number is added to each side of an equation.
(2) the same real number is subtracted from each side of an equation.
(3) each side of an equation is multiplied by the same non-zero real number.
(4) each side of an equation is divided by the same non-zero real number.

Do you think that the operation of taking the principal square root of each side of an equation will produce an equivalent equation? For example, consider the quadratic equation

$$x^2 = 25 \quad (1).$$

If we were to take the principal square root of each side of this equation, we would obtain the equation

$$\sqrt{x^2} = \sqrt{25}.$$

By the definition of principal square root, we know that $\sqrt{x^2} = |x|$ and $\sqrt{25} = 5$. Therefore, the above equation can be rewritten as

$$|x| = 5 \quad (2).$$

If equations (1) and (2) are equivalent, then they must have the same solution set. By inspection, we can tell that the solution set of $x^2 = 25$ is $\{5, -5\}$ since 5 and -5 are the only two real numbers whose square is 25. The solution set of the equation $|x| = 5$ is $\{5, -5\}$ since 5 and -5 are the only two real numbers whose absolute value is 5. Therefore, $x^2 = 25$ and $|x| = 5$ are equivalent equations since they both have the solution set $\{-5, 5\}$. In this case, the operation of taking the principal square root of each side of the equation $x^2 = 25$ produces the equivalent equation $|x| = 5$.

> In general, if the principal square root of each side of an equation is taken, an equivalent equation is obtained.

This new principle of equality, along with the others you have already learned, can be used to find the solution set of any quadratic equation that can be put in the form $ax^2 = c$ where $a, c \in R$.

EXAMPLE: Find the solution set of the quadratic equation $5x^2 - 30 = 0$, where $x \in R$.

Solution

$$5x^2 - 30 = 0$$

Add 30 to each side.

$$5x^2 = 30$$

Divide each side by 5.

$$x^2 = 6$$

Take the principal square root of each side.

$$\sqrt{x^2} = \sqrt{6}$$

$$|x| = \sqrt{6}$$

$$x = \pm\sqrt{6}$$

The solution set is $\{\sqrt{6}, -\sqrt{6}\}$.

In solving equations of the form $ax^2 = c$, you will sometimes find that the elements of the solution set are not real numbers. In these cases, if the domain of the variable is R, then the solution set must be ϕ, the empty set.

EXAMPLE: Find the solution set of the equation $-2x^2 = 8$, where $x \in R$.

Solution

$$-2x^2 = 8$$

Divide each side by -2.

$$x^2 = -4$$

Take the principal square root of each side.

$$\sqrt{x^2} = \sqrt{-4}$$

$$|x| = \sqrt{-4}$$

$$x = \pm\sqrt{-4}$$

Since $\sqrt{-4}$ and $-\sqrt{-4}$ are not real numbers, the solution set of this equation does not contain any real numbers. The solution set is ϕ.

In solving quadratic equations of the form $ax^2 = c$, where $a, c \in R$, you will find that if "a" and "c" have like signs, the equation has two roots that are equal in absolute value but opposite in sign.

EXAMPLES:

"a" AND "c" ARE BOTH POSITIVE.

$$3x^2 = 27$$

$$x^2 = 9$$

$$x = \pm 3$$

"a" AND "c" ARE BOTH NEGATIVE.

$$-5x^2 = -1$$

$$x^2 = \frac{1}{5}$$

$$x = \pm\sqrt{\frac{1}{5}}$$

NOT IN SIMPLEST FORM SINCE THE RADICAND IS A FRACTION.

$$x = \frac{\pm 1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$x = \frac{\pm\sqrt{5}}{5}$$

If "a" and "c" have unlike signs, the equation has no real roots.

EXAMPLES:

"a" IS NEGATIVE AND "c" IS POSITIVE.

$$\begin{aligned} -6x^2 &= 96 \\ x^2 &= -16 \\ x &= \pm\sqrt{-16} \end{aligned}$$

(not real)

"a" IS POSITIVE AND "c" IS NEGATIVE.

$$\begin{aligned} 4x^2 &= -12 \\ x^2 &= -3 \\ x &= \pm\sqrt{-3} \end{aligned}$$

(not real)

Remember that this method of solving equations can always be applied to quadratic equations which contain only second degree and constant terms. If a first degree term (i.e. a term in which x is to the first power) is also involved, this method does not apply.

Self-correcting Exercise #10

Answers may be found on page 67 of this lesson.

1. Find the solution set of each of the following equations. The domain of each variable is R.

(a) $x^2 = 16$

(b) $x^2 + 5 = 0$

(c) $16x^2 - 1 = 0$

(d) $1 - 3x^2 = 0$

(e) $-25x^2 = -1$

(f) $3x^2 + 3 = 2x^2 + 12$

B. Solving Quadratic Equations Which Can Be Put in the Form $ax^2 + bx = 0$, where $a, b \in R$

In order to solve quadratic equations of the form $ax^2 + bx = 0$ where $a, b \in R$, we must use the following property of real numbers.

> For any real numbers a and b, if $ab = 0$, then $a = 0$ or $b = 0$ or both.

This property tells us that if the product of two factors is zero, then at least one of the factors must be zero.

EXAMPLE: Find the solution set of the quadratic equation $2x^2 = 10x$ where $x \in R$.

Solution

$$2x^2 = 10x$$

Subtract 10x from both sides.

$$2x^2 - 10x = 0$$

Factor the left member. (Note that 2x is a common factor of both terms on the left.)

$$2x(x - 5) = 0$$

Since the product of the factors 2x and (x - 5) is zero, at least one of these factors must equal 0.

$$\text{i.e.}\quad 2x = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = \frac{0}{2} \qquad x = 5$$

$$x = 0$$

The solution set is $\{0, 5\}$.

In solving equations of the form $ax^2 + bx = 0$, you will find that such equations always have two real roots and one of these roots is always zero. Remember that this method of solving equations can always be applied to quadratic equations which contain only second degree and first degree terms.

Self-correcting Exercise #11

Answers may be found on page 68 of this lesson.

1. Find the solution set of each of the following equations. The domain of each variable is R.

   (a) $t^2 = 3t$ (b) $7y = y^2 - 2y$

(c) $-3y^2 + 12y = 0$ (d) $3x^2 + 5x = x^2 - 3x$

C. Using Factoring to Solve Quadratic Equations Which Can Be Put in the Form $ax^2 + bx + c = 0$ where $a, b, c \in R$

In a previous lesson, you learned that some polynomials of the form $ax^2 + bx + c$ where $a, b, c \in R$ can be expressed as the product of two binomials. For example,

$$\begin{aligned} x^2 - 3x - 10 &= (x - 5)(x + 2) \\ 12x^2 - 7x - 10 &= (3x + 2)(4x - 5) \\ -2x^2 - x + 3 &= -(2x + 3)(x - 1) \end{aligned}$$

If a quadratic equation can be put in the form $ax^2 + bx + c = 0$ and the left-hand side can be expressed as the product of two binomial factors, the solution set of the equation can be determined by setting each factor equal to zero and solving for x.

EXAMPLE: Find the solution set of the quadratic equation $12x^2 = -8x + 15$, where $x \in R$.

Solution

$$12x^2 = -8x + 15$$

Add 8x to each side.

$$12x^2 + 8x = 15$$

Subtract 15 from each side.

$$12x^2 + 8x - 15 = 0$$

Note that the left-hand side of this equation can be written as the product of the binomial factors $(6x - 5)$ and $(2x + 3)$.

$$(6x - 5)(2x + 3) = 0$$

Since the product of the factors $(6x - 5)$ and $(2x + 3)$ is zero, at least one of the factors must equal zero.

i.e. $6x - 5 = 0$ or $2x + 3 = 0$

$$6x = 5 \qquad 2x = -3$$

$$x = \frac{5}{6} \qquad x = \frac{-3}{2}$$

The solution set is $\left\{\frac{5}{6}, \frac{-3}{2}\right\}$.

Once you have put an equation in the form $ax^2 + bx + c = 0$, you should check the terms on the left side for common numerical factors before trying to factor it. If you can divide every term by the H.C.F., it will make factoring easier.

EXAMPLE: Find the solution set of $12x^2 + 56x = 20$ where $x \in R$.

Solution

$$12x^2 + 56x = 20$$

$$12x^2 + 56x - 20 = 0$$

Note that the three terms on the left have an H.C.F. of 4. Divide every term in the equation by 4.

$$\frac{12x^2}{4} + \frac{56x}{4} - \frac{20}{4} = \boxed{\frac{0}{4}} \leftarrow \text{RECALL THAT } 0/a = 0.$$

$$3x^2 + 14x - 5 = 0$$

Now, factor the left member and proceed as before.

$$(3x - 1)(x + 5) = 0$$

If $3x - 1 = 0$ or If $x + 5 = 0$

$3x = 1$ $\qquad x = -5$

$x = \frac{1}{3}$

The solution set is $\left\{\frac{1}{3}, -5\right\}$.

Self-correcting Exercise #12

Answers may be found on page 68 of this lesson.

1. Find the solution set of each of the following equations. The domain of each variable is R.

(a) $(x + 7)(x - 2) = 0$

(b) $(2x - 1)(3x + 7) = 0$

(c) $-(5x + 2)(x - 3) = 0$

(d) $(4x + 9)(x + 6) = 0$

(e) $x^2 = -3x + 4$

(f) $6x^2 - 17x = -5$

(g) $2x = -8x^2 + 3$ (h) $24x^2 = 54x + 15$

D. Using the Quadratic Formula to Solve Quadratic Equations Which Can Be Put in the Form $ax^2 + bx + c = 0$ where $a, b, c \in R$

Quadratic equations of the form $ax^2 + bx + c$ can be solved by factoring only if the binomial factors of $ax^2 + bx + c$ can be determined by inspection. We must try to find some general way of solving any quadratic equation in one variable. In order to solve for x in the general equation $ax^2 + bx + c = 0$, we would proceed as follows:

$$ax^2 + bx + c = 0$$

Subtract "c" from each side.

$$ax^2 + bx = -c$$

Divide every term by "a".

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

If we can make the left side of the above equation into a perfect square trinomial, then we can take the principal square root of each side of the equation. Note that $x^2 + \frac{b}{a}x$ will be a perfect square if we add $\frac{b^2}{4a^2}$ to it. $\left[\text{i.e. } x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2\right]$

Add $\frac{b^2}{4a^2}$ to both sides of the equation.

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$

SAME AMOUNT HAS BEEN ADDED TO BOTH SIDES.

Factor the left side of the equation.
(i.e. express it as the square of a binomial.)

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

Take the principal square root of each side.

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{-4ac + b^2}{4a^2}}$$

$$\left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Therefore, the solution set of the quadratic equation $ax^2 + bx + c = 0$ is $\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$.

The two roots of the equation are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$. These two roots are often written in combined form as $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (Note that the symbol "±" is read "plus or minus".) This expression is called the QUADRATIC FORMULA and should be memorized since it can be used to solve any quadratic equation in one variable.

Once a quadratic equation is put in the form $ax^2 + bx + c = 0$, the values of a, b, and c can easily be determined, and then these values can be substituted in the quadratic formula.

QUADRATIC FORMULA

The roots of any quadratic equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ can be determined by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

MEMORIZE.

Remember that this formula can only be applied to equations which are in the form $ax^2 + bx + c = 0$. Once an equation is in this form, the values of a, b, and c can be determined by analyzing the terms on the left side of the equation.

"a" is the numerical coefficient of the second degree term.
"b" is the numerical coefficient of the first degree term.
"c" is the constant term.

EXAMPLE: Solve the quadratic equation $2x^2 = 4x + 7$, where $x \in R$.

Solution

First, put the equation in the form $ax^2 + bx + c = 0$.

$$2x^2 = 4x + 7$$

$$2x^2 - 4x - 7 = 0$$

Determine the values of a, b, and c.

$a = 2$ since the coefficient of x^2 is 2.
$b = -4$ since the coefficient of x is -4.
$c = -7$ since the constant term is -7.

Now substitute these values for a, b, and c into the quadratic formula.
($a = 2$, $b = -4$, $c = -7$)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-7)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{16 + 56}}{4}$$

$$= \frac{4 \pm \sqrt{72}}{4}$$

An approximation to $\sqrt{72}$ can be found in the table of square roots on page 57 of lesson B.

$$= \frac{4 \pm 8.485}{4}$$

$$= \frac{4 + 8.485}{4} \quad \text{or} \quad \frac{4 - 8.485}{4}$$

$$= 3.121 \quad \text{or} \quad -1.121$$

The solution set is $\{3.121, -1.121\}$.

Use the quadratic formula to solve the equation $x^2 - 5x + 3 = 0$.
In this case, $a = 1$, $b = -5$, $c = \underline{\qquad}$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(\quad) \pm \sqrt{(\quad)^2 - 4(\quad)(\quad)}}{2(\underline{\quad})}$$

$$= \frac{\qquad \pm \sqrt{25 - \qquad}}{2}$$

$$= \frac{\qquad \pm \sqrt{\qquad}}{2}$$

USE SQUARE ROOT TABLE IN LESSON B TO EVALUATE THIS ROOT.

$$= \frac{\qquad \pm \qquad}{2}$$

$$= \frac{\qquad}{2} \quad \text{or} \quad \frac{\qquad}{2}$$

$$= \underline{\qquad} \quad \text{or} \quad \underline{\qquad}$$

The solution set is $\underline{\qquad\qquad}$.

Once you have put an equation in the form $ax^2 + bx + c = 0$, you should check the terms on the left side for common numerical factors before applying the quadratic formula. Your work will be simplified considerably if you divide every term by the highest common factor before evaluating a, b, and c.

EXAMPLE: Solve the quadratic equation $18x = -9x^2 - 36$, where $x \in R$.

Solution $18x = -9x^2 - 36$

$9x^2 + 18x + 36 = 0$

Note that the H.C.F. of the 3 terms on the left is 9.

Divide every term in the equation by 9.

$$\frac{9x^2}{9} + \frac{18x}{9} + \frac{36}{9} = \frac{0}{9}$$

$$x^2 + 2x + 4 = 0$$

In this case, $a = 1$, $b = 2$, and $c = 4$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} = \frac{-2 \pm \sqrt{-12}}{2}$$

The roots are $\frac{-2 + \sqrt{-12}}{2}$ and $\frac{-2 - \sqrt{-12}}{2}$ which are not real numbers.

Therefore the solution set is ϕ when the domain of x is R.

Self-correcting Exercise #13

Answers may be found on page 69 of this lesson.

1. Fill in the blanks in the following chart. (Domain of each variable is R.)

	$y^2 + 8y = 2$	$5x + 1 = -2x^2$	$x^2 = 6x - 12$
Put equation in form $ax^2 + bx + c = 0$	$y^2 + 8y - 2 = 0$		
State the values of a, b, and c.	$a = 1$, $b = 8$, $c = -2$		
Apply quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(-2)}}{2(1)}$ $=$		
State the roots.			
State solution set.			

2. Use the quadratic formula to find the roots of the following equations.

(a) $3x^2 + 6x - 1 = 0, x \in R$ (b) $3x^2 = 5(x - 1)^2, x \in R$

E. Summary - Solving Quadratic Equations in One Variable

The quadratic formula enables us to solve any quadratic equation in one variable. But, some of these equations are more easily solved by other methods. For example, the equation $5x^2 - 7x = 0$ can be solved by applying the quadratic formula (see Method II below), but it can be solved more efficiently by factoring the left member, setting each factor equal to zero, and then solving for x (see Method I below).

METHOD I

$$5x^2 - 7x = 0$$
$$x(5x - 7) = 0$$
$$x = 0 \text{ or } 5x - 7 = 0$$
$$5x = 7$$
$$x = \frac{7}{5}$$

METHOD II

$$5x^2 - 7x = 0$$
$$a = 5,\ b = -7,\ c = 0$$
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(5)(0)}}{2(5)} \\ &= \frac{7 \pm \sqrt{49 - 0}}{10} \\ &= \frac{7 \pm 7}{10} \\ &= \frac{7}{5} \text{ or } 0 \end{aligned}$$

Doesn't Method I seem to be much simpler?

Self-correcting Exercise #14

Answers may be found on page 70 of this lesson.

1. Solve each of the following quadratic equations. The domain of each variable is R. Try to use the simplest method of solution possible.

(a) $3y^2 = 25$

(b) $x^2 = 13x - 36$

(c) $3x = 5x^2$

(d) $4x^2 + 5 = 9$

(e) $2x^2 - 6x - 1 = 0$

(f) $3x^2 - 10x + 2 = 0$

F. Nature of the Roots of a Quadratic Equation

When you are finding the roots of a quadratic equation in one variable, exactly one of the following three possibilities will occur.

1. There are two roots which are real numbers and have different values.
2. There are two roots which are real numbers and have the same value.
3. There are two roots which are not real numbers.

These three possibilities are illustrated in the examples below.

EXAMPLE 1

$x^2 + 3x + 1 = 0$

$a = 1,\ b = 3,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{9 - 4}}{2}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

Root are $\frac{-3 + \sqrt{5}}{2}$ and $\frac{-3 - \sqrt{5}}{2}$.

There are two roots which are real and distinct.

EXAMPLE 2

$x^2 - 2\sqrt{5}x + 5 = 0$

$a = 1,\ b = -2\sqrt{5},\ c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2\sqrt{5} \pm \sqrt{(-2\sqrt{5})^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{2\sqrt{5} \pm \sqrt{20 - 20}}{2}$$

$$= \frac{2\sqrt{5} \pm 0}{2}$$

Roots are $\sqrt{5}$ and $\sqrt{5}$.

There are two roots which are real and equal.

EXAMPLE 3

$x^2 - 5x + 8 = 0$

$a = 1,\ b = -5,\ c = 8$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25 - 32}}{2}$$

$$= \frac{5 \pm \sqrt{-7}}{2}$$

Roots are $\frac{5 + \sqrt{-7}}{2}$ and $\frac{5 - \sqrt{-7}}{2}$.

There are two roots which are not real numbers.

In Example 1 above, note that the expression under the root sign is equal to 5, a positive number, and the resulting roots are real and distinct. In Example 2, the expression under the root sign is equal to zero and the resulting roots are real and equal. In Example 3, the expression under the root sign is negative and the roots are non-real.

In general, for the quadratic equation $ax^2 + bx + c = 0$, the roots are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

(1) If $b^2 - 4ac > 0$, $\sqrt{b^2 - 4ac}$ will be a positive real number. The two roots of the equation will have different values since $\sqrt{b^2 - 4ac}$ is added to find one root and subtracted to find the other. Thus, if $b^2 - 4ac > 0$, then $ax^2 + bx + c = 0$ has two real and distinct roots.

(2) If $b^2 - 4ac = 0$, the two roots will be $\frac{-b + 0}{2a}$ and $\frac{-b - 0}{2a}$. Each of these roots is equal to $\frac{-b}{2a}$. Thus, if $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has two real and equal roots.

(3) If $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac}$ is not a real number and the roots will not be real numbers. Thus, if $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has no real roots.

Because the value of $b^2 - 4ac$ determines the nature of the roots, it enables us to distinguish or discriminate amongst these three possibilities. For this reason, the expression $b^2 - 4ac$ is called the DISCRIMINANT of the quadratic equation $ax^2 + bx + c = 0$. By evaluating the discriminant of a quadratic equation, we can comment on the nature of the roots of the equation without actually evaluating the roots.

NATURE OF THE ROOTS OF $ax^2 + bx + c = 0$

If $b^2 - 4ac > 0$, there are two real and distinct roots.
If $b^2 - 4ac = 0$, there are two real and equal roots.
If $b^2 - 4ac < 0$, there are no real roots.

EXAMPLE 1: Describe the nature of the roots of the quadratic equations $x^2 - 3x = 0$, $x^2 + 3x = -5$, and $9x^2 = 30x - 25$.

Solution

$x^2 - 3x = 0$
$a = 1,\ b = -3,\ c = 0$
$b^2 - 4ac$
$= (-3)^2 - 4(1)(0)$
$= 9 - 0$
$= 9$

Since $b^2 - 4ac > 0$, the roots are real and distinct.

$x^2 + 3x = -5$
$x^2 + 3x + 5 = 0$
$a = 1,\ b = 3,\ c = 5$
$b^2 - 4ac$
$= (3)^2 - 4(1)(5)$
$= 9 - 20$
$= -11$

Since $b^2 - 4ac < 0$, there are no real roots.

$9x^2 = 30x - 25$
$9x^2 - 30x + 25 = 0$
$a = 9,\ b = -30,\ c = 25$
$b^2 - 4ac$
$= (-30)^2 - 4(9)(25)$
$= 900 - 900$
$= 0$

Since $b^2 - 4ac = 0$, the roots are real and equal.

Self-correcting Exercise #15

Answers may be found on page 71 of this lesson.

1. Evaluate the discriminant of each of the following equations and describe the nature of the roots.

(a) $4x^2 - x - 5 = 0$

(b) $9x^2 + 12x + 4 = 0$

(c) $z^2 = -z - 1$

2. For what real values of k does the quadratic equation $kx^2 - 2x + 1 = 0$ have two real and distinct roots?

EXERCISE - Quadratic Equations in One Variable

1. Fill in the blanks.

(a) If $x \in R$, the solution set of the equation $x^2 = -36$ is ________.

(b) If $|x| = 7$, then we know that $x =$ ________ or $x =$ ________.

(c) If $x \in R$, the solution set of the equation $x^2 + 5x = 0$ is ________.

(d) Before the quadratic formula can be used to solve the equation $x^2 = -3x + 5$, this equation must be put in the form ________________.

(e) $b^2 - 4ac$ is called the ________________ of the quadratic equation $ax^2 + bx + c = 0$.

(f) The equations $x^2 = 64$ and $|x| = 8$ are ____________________________ equations because they both have the solution set ________________.

(g) All equations of the form $ax^2 + bx = 0$ have two roots, one of which is equal to __________.

(h) The roots of the equation $x^2 - 6x + 9 = 0$ are real and __________.

(i) Any quadratic equation of the form $ax^2 = c$, where a and c are real numbers with like signs, has two real roots that are equal in ______________________ but opposite in ________________.

2. Find the roots and state the solution set of each of the following equations. The domain of each variable is R. Try to use the simplest method of solution possible.

(a) $9x^2 - 16 = 0$

Solution set is __________.

(b) $x^2 + 8x = -15$

Solution set is __________.

(c) $x^2 + 4x = 3$

Solution set is __________.

(d) $6m^2 + m = 0$

Solution set is __________.

(e) $2x^2 + 9 = 0$

Solution set is __________.

(f) $2x^2 = 3 - 5x$

Solution set is __________.

(g) $x^2 + 25 = 10x$

Solution set is __________.

(h) $x^2 = 11 - x$

Solution set is __________.

(i) $x^2 + x + 2 = 0$

Solution set is __________.

(j) $4m = m^2$

Solution set is __________.

3. Evaluate the discriminant and describe the nature of the roots of each of the following equations.

(a) $x^2 = -2x + 3$

$x^2 + 2x - 3 = 0$

$b^2 - 4ac =$

Roots are _______________.

(b) $-x^2 + x = 1$

Roots are _______________.

(c) $x^2 - 3\sqrt{2}x + 5 = 0$

Roots are _______________.

(d) $4x^2 + 4x = -1$

Roots are _______________.

4. For what real values of m does the quadratic equation $mx^2 - 8x - 9 = 0$ have no real roots? (See page 47.)

5. Decide whether each of the following statements is true or false.

True or False?

(a) The solution set of $x^2 + 9 = 0$ is $\{-3, 3\}$. ____________

(b) The solution set of $3x^2 - 12 = 0$ is $\{2\}$. ____________

(c) $\{x \mid (x + 2)^2 = 0\} = \{-2\}$. ____________

(d) If $\{x \mid 2x^2 + x - 6 = 0\} = \{x \mid x^2 + bx - 3 = 0\}$, then $b = \frac{1}{2}$. ____________

(e) If $\{x \mid (x - 4)(x + a) = 0\} = \{4, 2\}$, then $a = 2$. ____________

(f) If $(x + 2)^2 = 12$, then $x + 2 = \sqrt{12}$ or $x + 2 = -\sqrt{12}$. ____________

(g) For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac < 0$, there are no real roots. ____________

(h) The discriminant of $x^2 + 4x - 3 = 0$ is 4. ____________

(i) The equation $x^2 - 7x + 12 = 0$ can only be solved by using the quadratic formula. ____________

(j) $\frac{-6 + \sqrt{8}}{10} = \frac{-3 + \sqrt{2}}{5}$ ____________

(k) The equation $x^2 + 2x + 1 = 0$ has two real and distinct roots. ____________

6. Form a quadratic equation that has roots of 7 and 3.

$$(x \qquad)(x \qquad) = 0$$

$$x^2 - \underline{\quad} + \underline{\quad} = 0$$

7. For what values of k will the equation $2x^2 + 4x + (2 - k - k^2) = 0$ have two real and equal roots?

8. The length of a rectangle plus its width is 13 units. The area is 40 square units. Determine the length and width.

Solution

Let the width be x units.
Then the length is $(13 - x)$ units.

$$\text{Length} \times \text{Width} = \text{Area}$$

$$(\qquad\qquad)x = \underline{\qquad}$$

$$\left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{Solve for } x.$$

The length is ______ units and the width is ______ units.

Topic Four: Fractional and Radical Equations in One Variable

A. Fractional Equations

A FRACTIONAL EQUATION is an equation which has a variable in the denominator of one or more of its terms. The following are examples of fractional equations.

$$(1)\quad \frac{4x+2}{x^2-1} - \frac{1}{x+1} = \frac{1}{x-1}$$

$$(2)\quad \frac{2x+1}{x-3} = \frac{x+4}{x} + \frac{8}{5}$$

Before trying to solve a fractional equation, you must state any values of the variable for which a denominator is zero. Since division by zero is impossible, these values are not in the domain of the variable and cannot belong to the solution set of the equation. In equation (1) above, $x \neq 1$ and $x \neq -1$ since these values of x make at least one of the denominators zero. In equation (2) above $x \neq 3$ and $x \neq 0$.

A fractional equation can be transformed into an equation that is not fractional by multiplying every term by the L.C.D. of all the terms in the equation. Once the equation has been cleared of fractions, it can often be solved as a linear or quadratic equation in one variable.

Note how equations (1) and (2) above can be solved.

EQUATION (1): $$\frac{4x+2}{x^2-1} - \frac{1}{x+1} = \frac{1}{x-1}, \quad \text{where } x \neq 1 \text{ and } x \neq -1.$$

$$\frac{4x+2}{(x+1)(x-1)} - \frac{1}{x+1} = \frac{1}{x-1}$$

Multiply every term by the L.C.D., $(x+1)(x-1)$.
(See boxed-in sections below.)

$$\boxed{(x+1)(x-1)}\frac{4x+2}{(x+1)(x-1)} - \boxed{(x+1)(x-1)}\frac{1}{(x+1)} = \boxed{(x+1)(x-1)}\frac{1}{(x-1)}$$

Cancel like factors in the numerator and denominator of each term.

$$\frac{\cancel{(x+1)}\cancel{(x-1)}(4x+2)}{\cancel{(x+1)}\cancel{(x-1)}} - \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}} = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$(4x+2) \quad - \quad (x-1) \quad = (x+1)$$

Note that the equation is now a linear equation in one variable.

$$\begin{aligned} 4x + 2 - x + 1 &= x + 1 \\ 3x + 3 &= x + 1 \\ 2x + 3 &= 1 \\ 2x &= -2 \\ x &= -1 \end{aligned}$$

But, since $x = -1$ is not in the domain of x, there is no root and the solution set is $\not\phi$.

EQUATION (2): $\frac{2x + 1}{x - 3} = \frac{x + 4}{x} + \frac{8}{5}$, $x \neq 3$ and $x \neq 0$

Multiply every term by the L.C.D., $5x(x - 3)$.

$$5x(x - 3) \cdot \frac{2x + 1}{x - 3} = 5x(x - 3) \cdot \frac{x + 4}{x} + 5x(x - 3) \cdot \frac{8}{5}$$

Cancel like factors in the numerator and denominator of each term.

$$\frac{5x\cancel{(x - 3)}(2x + 1)}{\cancel{(x - 3)}} = \frac{5\cancel{x}(x - 3)(x + 4)}{\cancel{x}} + \frac{\cancel{5}x(x - 3)8}{\cancel{5}}$$

$$\begin{aligned} 5x(2x + 1) &= 5(x - 3)(x + 4) + 8x(x - 3) \\ 10x^2 + 5x &= 5(x^2 + x - 12) + 8x^2 - 24x \\ 10x^2 + 5x &= 5x^2 + 5x - 60 + 8x^2 - 24x \\ 10x^2 + 5x &= 13x^2 - 19x - 60 \\ -3x^2 + 24x + 60 &= 0 \end{aligned}$$

Divide every term by -3.

$$x^2 - 8x - 20 = 0$$

Note that the equation is now a quadratic equation in one variable.

$$(x - 10)(x + 2) = 0$$

If $x - 10 = 0$, $x = 10$ or If $x + 2 = 0$, $x = -2$

Since 10 and -2 are among the permissible values of x, they must belong to the solution set.

The solution set is $\{10, -2\}$.

Fill in the blanks in order to solve the two fractional equations below.

$$\frac{1}{x} + \frac{x-1}{x(x+2)} = \frac{-x}{x+2}$$

$$x \neq \underline{\qquad}, \quad x \neq \underline{\qquad}$$

Multiply every term by x(x + 2).

$$x(x+2)\,\frac{1}{x} + \underline{\qquad}\,\frac{x-1}{x(x+2)} = \underline{\qquad}\,\frac{-x}{x+2}$$

Cancel like factors where possible.

$$\frac{\overset{1}{\cancel{x}}(x+2)}{\underset{1}{\cancel{x}}} + \frac{(x-1)}{x(x+2)} = \frac{(-x)}{(x+2)}$$

$$(x+2) + (\underline{\qquad}) = \underline{\qquad}(-x)$$

$$2x + \underline{\qquad} = \underline{\qquad}$$

$$\underline{\qquad} + 2x + \underline{\qquad} = 0$$

$$(\underline{\qquad})(\underline{\qquad}) = 0$$

$$x = \underline{\qquad}$$

Is this value among the permissible values of x? $\underline{\qquad\qquad}$.

Solution set is $\underline{\qquad\qquad}$.

$$\frac{3x^2}{3x+1} - 2 = \frac{2x+1}{3x+1}, \quad x \neq \underline{\qquad}$$

Multiply every term by ($\underline{\qquad\qquad}$).

$$(3x+1)\,\frac{\qquad}{3x+1} - 2(3x+1) = (\underline{\qquad})\frac{\qquad}{3x+1}$$

Cancel like factors where possible.

$$\frac{\overset{1}{\cancel{(3x+1)}}(\quad)}{\underset{1}{\cancel{(3x+1)}}} - 2(3x+1) = \frac{(\qquad)(\qquad)}{(3x+1)}$$

$$\underline{\qquad} - 6x - 2 = \underline{\qquad} + \underline{\qquad}$$

$$\underline{\qquad\qquad\qquad} = 0$$

$$(\qquad)(\qquad) = 0$$

If $\underline{\qquad} = 0$ If $\underline{\qquad} = 0$

$x = \underline{\qquad}$ $x = \underline{\qquad}$

Which of these roots is not among the permissible values of x? $\underline{\qquad\qquad}$

Solution set is $\underline{\qquad\qquad}$.

B. Radical Equations

A RADICAL EQUATION is an equation in which the variable occurs in a radicand. The following are examples of radical equations.

(1) $\sqrt{x+5} = 3$
(2) $3 = 2\sqrt{x} + x$

When solving radical equations, we must use the process of squaring both sides of the equation in order to remove the variable expression from under the root sign. This squaring process does not always yield an equivalent equation but it does yield an equation whose roots have the roots of the original equation among them. That is, the solution set of the original equation will always be a subset of the solution set of the new equation that was formed by the squaring process.

For example, if we are given the equation $x = 5$ and square both sides of this equation, we obtain the new equation $x^2 = 25$. These two equations are not equivalent since the solution set of $x = 5$ is $\{5\}$ and the solution set of $x^2 = 25$ is $\{-5, 5\}$. But, note that the root of $x = 5$ is among the roots of $x^2 = 25$. On the other hand, if we are given the equation $\sqrt{x} = 3$ and square both sides of the equation, we obtain the new equation $x = 9$. These equations are equivalent since they both have a solution set of $\{9\}$.

Therefore, if we use the squaring process in solving equations, we cannot be sure that we will obtain equivalent equations. But, we can be sure that the roots of the original equation will be among the roots of the new equation formed by the squaring process.

Once we have determined the roots of the new equation, we must check each of these roots by substitution in the original equation. Each root which satisfies the original equation will belong to the solution set of that equation.

In solving a radical equation which contains only one radical term, we must isolate this term on one side of the equation. Then, square both sides of the equation in order to remove the variable expression from under the root sign.

EXAMPLE: Find the solution set of the equation $\sqrt{4 - 3x} - x = 12$.

Solution

$$\sqrt{4 - 3x} - x = 12$$

Isolate the radical term on the left-hand side.

$$\sqrt{4 - 3x} = x + 12$$

Square both sides of the equation.

$$(\sqrt{4 - 3x})^2 = (x + 12)^2$$

$$4 - 3x = x^2 + 24x + 144$$

$$-x^2 - 27x - 140 = 0$$

Multiply every term by -1.

$$x^2 + 27x + 140 = 0$$

$$(x + 7)(x + 20) = 0$$

$$x + 7 = 0 \quad \text{or} \quad x + 20 = 0$$

$$x = -7 \qquad\qquad x = -20$$

Thus, -7 and -20 are the possible roots of the original equation. We must check these roots by substitution.

Check

If $x = -7$,

$\sqrt{4 - 3x} - x$	12
$\sqrt{4 - 3(-7)} - (-7)$	
$\sqrt{4 + 21} + 7$	
$\sqrt{25} + 7$	
$5 + 7$	
12	

-7 is a root

If $x = -20$,

$\sqrt{4 - 3x} - x$	12
$\sqrt{4 - 3(-20)} - (-20)$	
$\sqrt{4 + 60} + 20$	
$\sqrt{64} + 20$	
$8 + 20$	
28	

-20 is not a root.

Conclusion

The solution set of the equation $\sqrt{4 - 3x} - x = 12$ is $\{-7\}$.

Solve the following radical equation and then check your solution.

$$x = 2 - \sqrt{2x - 5}$$

Isolate the radical term on the right side.

$x - \underline{\quad\quad} = -\sqrt{2x - 5}$

Square both sides.

$(x - \underline{\quad\quad})^2 = (-\sqrt{2x - 5})^2$

$x^2 - \underline{\quad\quad} + \underline{\quad\quad} = 2x - \underline{\quad\quad}$

$x^2 - \underline{\quad\quad} + \underline{\quad\quad} = 0$

$(x \quad\quad)(x \quad\quad) = 0$

$x = \underline{\quad\quad}$

$x = \underline{\quad\quad\quad}$ is a possible root of the original equation.

Check

If $x = \underline{\quad\quad\quad}$,

x	$2 - \sqrt{2x - 5}$
$\underline{\quad\quad}$	$2 - \sqrt{2(\underline{\quad}) - 5}$
	$2 - \sqrt{\underline{\quad\quad} - 5}$
	$2 - \sqrt{\underline{\quad\quad}}$
	$2 - \underline{\quad\quad}$
	$\underline{\quad\quad\quad}$

Is this value of x a root? $\underline{\quad\quad\quad}$

The solution set is $\underline{\quad\quad\quad\quad}$.

When solving an equation which contains two radical terms, you must write the equation with a radical term on each side.

EXAMPLE: Find the solution set of the equation $\sqrt{x - 6} - \sqrt{x + 9} = -3$.

Solution

$$\sqrt{x - 6} - \sqrt{x + 9} = -3$$

Rewrite the equation so that there is a radical term on each side.

$$\sqrt{x - 6} = \sqrt{x + 9} - 3$$

Now, square both sides.

SEE PAGES 49 AND 50, LESSON D, ON SQUARING A BINOMIAL.

$$(\sqrt{x - 6})^2 = (\sqrt{x + 9} - 3)^2$$
$$x - 6 = (\sqrt{x + 9})^2 + 2(\sqrt{x + 9})(-3) + (-3)^2$$
$$x - 6 = x + 9 - 6\sqrt{x + 9} + 9$$
$$x - 6 = x - 6\sqrt{x + 9} + 18$$

Since we still have a radical term, this term must be isolated on one side of the equation.

$$6\sqrt{x + 9} = 24$$

Divide each side of the equation by 6.

$$\sqrt{x + 9} = 4$$

Square both sides of the equation.

$$(\sqrt{x + 9})^2 = (4)^2$$
$$x + 9 = 16$$
$$x = 7$$

We must check this root in the original equation.

Check

If $x = 7$,

$\sqrt{x - 6} - \sqrt{x + 9}$	-3
$\sqrt{7 - 6} - \sqrt{7 + 9}$	
$\sqrt{1} - \sqrt{16}$	
$1 - 4$	
-3	

7 is a root.

Conclusion

The solution set of the equation $\sqrt{x - 6} - \sqrt{x + 9} = -3$ is $\{7\}$.

Solve the following radical equation and then check your solution.

$$\sqrt{2x+3} - \sqrt{x+1} = 1$$

$$\sqrt{2x+3} = \sqrt{\qquad} + 1$$

Square both sides.

$$(\sqrt{2x+3})^2 = (\sqrt{\qquad} + 1)^2$$

$$\underline{\quad} + \underline{\quad} = (\sqrt{\qquad})^2 + 2(\sqrt{\qquad})(1) + (1)^2$$

$$\underline{\quad} + \underline{\quad} = \underline{\quad} + \underline{\quad} + 2\sqrt{\qquad} + 1$$

$$\underline{\quad} + \underline{\quad} = x + 2 + 2\sqrt{\qquad}$$

$$\underline{\quad} + \underline{\quad} = 2\sqrt{\qquad}$$

Square both sides.

$$(x+1)^2 = (2\sqrt{x+1})^2$$

$$\underline{\qquad\qquad} = 4(x+1)$$

$$\underline{\qquad\qquad} = 4x + \underline{\quad}$$

$$\underline{\qquad\qquad} = 0$$

$$(\qquad)(\qquad) = 0$$

If $(\qquad) = 0$ If $(\qquad) = 0$

$x = \underline{\quad}$ $x = \underline{\quad}$

Check

If $x = \underline{\quad}$,

$\sqrt{2x+3} - \sqrt{x+1}$	1
$\sqrt{2(\quad)+3} - \sqrt{\underline{\quad}+1}$	
$\sqrt{\underline{\quad}+3} - \sqrt{\underline{\quad}}$	
$\sqrt{\underline{\quad}} - \sqrt{\underline{\quad}}$	
$\underline{\quad} - \underline{\quad}$	
$\underline{\qquad}$	

Is this value of x a root? ______

If $x = \underline{\quad}$,

$\sqrt{2x+3} - \sqrt{x+1}$	1
$\sqrt{2(\quad)+3} - \sqrt{\underline{\quad}+1}$	
$\sqrt{\underline{\quad}+3} - \sqrt{\underline{\quad}}$	
$\sqrt{\underline{\quad}} - \sqrt{\underline{\quad}}$	
$\underline{\quad} - \underline{\quad}$	
$\underline{\qquad}$	

Is this value of x a root? ______

Conclusion

The solution set of the equation $\sqrt{2x+3} - \sqrt{x+1} = 1$ is __________.

EXERCISE - Fractional and Radical Equations

Find the solution set of each of the following equations.

1. $\frac{x + 3}{x} - x = 5 + \frac{3}{x}$, $x \neq$ ____

Multiply every term by x.

Which of the roots is not among the permissible values of x? ________

Solution set is __________.

2. $\sqrt{2x - 3} + 4 = 9$

Isolate the radical on one side.

Check

If x = ____,

Solution set is ___________.

3. $1 + \frac{12}{x^2 - 4} = \frac{3}{x - 2}$, $x \neq$ ______ and $x \neq$ ______

Hint: (First, factor the denominator $x^2 - 4$.
The L.C.D. is $(x + 2)(x - 2)$.)

Which root is not among the permissible values of x? ____________

Solution set is ____________.

4. $\frac{3}{x - 3} + \frac{1}{x - 1} = \frac{x}{x^2 - 4x + 3}$, $x \neq$ ______ and $x \neq$ ______

Solution set is ______________.

5. $2 - y - 2\sqrt{y + 1} = 0$

Isolate the radical on one side.

Check

If $y =$ _____,

If $y =$ _____,

Solution set is __________.

6. $\sqrt{x + 2} = 1 + \sqrt{3 - x}$

Square both sides.

Check

If $x =$ _____,

If $x =$ _____,

Solution set is __________.

Key to Self-correcting Exercises in Lesson F

Exercise #1, page 2

1. (a) number; only numbers are involved.
   (b) conditional; variable x is involved.
   (c) number; only numbers are involved.
   (d) identity; true for any value of x.
   (e) conditional; variables x and y are involved.
   (f) identity; true for any values of a, b, and c.

2. (a) $14 - 5 = 9$ is a true statement.
   (b) $9 + 1 = 9$ is a false statement.
   (c) $3 + 9 = 1$ is a false statement.
   (d) $5 + 3 = 8$ is a true statement.
   (e) $18 - 2 = 7$ is a false statement.
   (f) $1(0 + 1) = (1)(0) + 1$ is a true statement.

Exercise #2, page 4

1. (a) quadratic in one variable
   (b) linear in one variable
   (c) quadratic in two variables
   (d) cubic in two variables
   (e) cubic in three variables
   (f) quadratic in one variable
   (g) quartic in two variables
   (h) quadratic in three variables
   (i) cubic in one variable
   (j) quadratic in two variables
   (k) quadratic in one variable
   (l) linear in two variables

Exercise #3, page 5

1. (a) $x + 4 = 9$ is satisfied when $x = 5$ since $5 + 4 = 9$ is a true statement.

   (b) $x - 3 = 7$ is satisifed when $x = 10$ since $10 - 3 = 7$ is a true statement.

   (c) $-3x = 21$ is satisified when $x = -7$ since $-3 \times -7 = 21$ is a true statement.

   (d) $15 - x = 6$ is satisfied when $x = 9$ since $15 - 9 = 6$ is a true statement.

   (e) $\frac{x}{-2} = 8$ is satisfied when $x = -16$ since $\frac{-16}{-2} = 8$ is a true statement.

2. (a) $x = 11$
   Solution set is $\{11\}$.

   (b) $x = 44$
   Solution set is $\{44\}$.

   (c) $x = 69$
   Solution set is $\{69\}$.

   (d) $x = 450$
   Solution set is $\{450\}$.

   (e) $x = 9$
   Solution set is $\{9\}$.

   (f) $x = 3$
   Solution set is $\{3\}$.

Exercise #4, page 9

1. (a) Divide each side by 2.
   (b) Subtract 3 from each side.
   (c) Add 2 to each side.
   (d) Multiply each side by -2.
   (e) Divide each side by $\frac{3}{4}$.
   (i.e. multiply each side by $\frac{4}{3}$.)

2. (a) $x - 7 = 3$
   $x = 3 + 7$
   $x = 10$
   Solution set is $\{10\}$.

   (b) $\frac{x}{4} = -1$
   $x = 4 \times -1$
   $x = -4$
   Solution set is $\{-4\}$.

   (c) $x + 9 = 34$
   $x = 34 - 9$
   $x = 25$
   Solution set is $\{25\}$.

   (d) $-5x = 35$
   $x = \frac{35}{-5}$
   $x = -7$
   Solution set is $\{-7\}$.

   (e) $\frac{3}{5}x = \frac{9}{25}$
   $x = \dfrac{\overset{3}{\cancel{9}}}{\underset{5}{\cancel{25}}} \times \dfrac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{3}}}$
   $x = \frac{3}{5}$
   Solution set is $\left\{\frac{3}{5}\right\}$.

   (f) $\frac{x}{5} = 0$
   $x = 0 \times 5$
   $x = 0$
   Solution set is $\{0\}$.

Exercise #5, page 13

1. (a) Subtract 7 from both sides.
   Divide both sides by 3.

   (b) Subtract 3x from both sides.
   Add 6 to both sides.
   Multiply both sides by -1.

   (c) Multiply every term by 6.
   Subtract x from both sides.
   Add 4 to both sides.
   Divide both sides by 2.

   (d) Collect like terms.
   Add x to both sides.
   Subtract 3 from both sides.
   Divide both sides by -3.

   (e) Apply distributive property.
   Collect like terms.
   Subtract 2x from both sides.
   Subtract 8 from both sides.
   Divide both sides by 2.

   (f) Multiply every term by 12.
   Apply distributive property.
   Collect like terms.
   Add 18x to both sides.
   Add 12 to both sides.
   Divide both sides by 21.

Exercise #5, continued

2. (a)

$$\begin{aligned} 2x - 8 &= 7 + 3x \\ -x - 8 &= 7 \\ -x &= 15 \\ x &= -15 \end{aligned}$$

Solution set is $\{-15\}$.

(b)

$$\begin{aligned} \frac{x}{5} + \frac{2}{15} &= \frac{1}{10} \\ \overset{6}{\cancel{30}}\left(\frac{x}{\underset{1}{\cancel{5}}}\right) + \overset{2}{\cancel{30}}\left(\frac{2}{\underset{1}{\cancel{15}}}\right) &= \overset{3}{\cancel{30}}\left(\frac{1}{\underset{1}{\cancel{10}}}\right) \\ 6x + 4 &= 3 \\ 6x &= -1 \\ x &= \frac{-1}{6} \end{aligned}$$

Solution set is $\left\{\frac{-1}{6}\right\}$.

(c)

$$\begin{aligned} 4(x - 2) &= 3(2x - 6) \\ 4x - 8 &= 6x - 18 \\ -2x - 8 &= -18 \\ -2x &= -10 \\ x &= 5 \end{aligned}$$

Solution set is $\{5\}$.

(d)

$$\begin{aligned} \frac{55y + 7}{6} &= 7 - \frac{23y - 1}{24} \\ \overset{4}{\cancel{24}}\left(\frac{55y + 7}{\underset{1}{\cancel{6}}}\right) &= (24 \times 7) - \overset{1}{\cancel{24}}\left(\frac{23y - 1}{\underset{1}{\cancel{24}}}\right) \\ 4(55y + 7) &= 168 - (23y - 1) \\ 220y + 28 &= 168 - 23y + 1 \\ 220y + 28 &= -23y + 169 \\ 243y + 28 &= 169 \\ 243y &= 141 \\ y &= \frac{47}{81} \end{aligned}$$

Solution set is $\left\{\frac{47}{81}\right\}$.

Exercise #6, page 21

1. (a) conditional (b) number (c) unconditional
   (d) conditional (e) number (f) unconditional

2. (a) $10 < 12$ is a true statement.
   (b) $4 + 1 < 3$ is a false statement.
   (c) $-5 < -4 + 1$ is a true statement.
   (d) $1 + 1 > 2$ is a false statement.
   (e) $16 + 100 > 25$ is a true statement.
   (f) $\frac{1}{4} \geqq \frac{1}{2}$ is a false statement.

Exercise #7, page 22

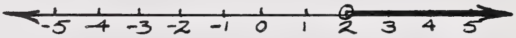
1. (a) quadratic in one variable (b) cubic in two variables
   (c) linear in one variable (d) quadratic in two variables
   (e) quadratic in one variable (f) linear in two variables
   (g) quadratic in two variables (h) cubic in one variable

Exercise #8, page 24

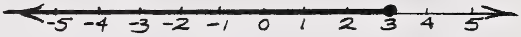
1. (a) The inequality $3x \leqq 15$ is satisfied when $x \leqq 5$ since it becomes a true statement when any real number less than or equal to 5 is substituted for x. The solution set is $\{x | x \leqq 5, x \in R\}$.

   (b) The inequality $x + 2 > 8$ is satisfied when $x > 6$ since it becomes a true statement when any real number greater than 6 is substituted for x. The solution set is $\{x | x > 6, x \in R\}$.

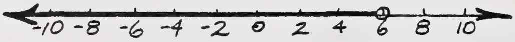
2. (a) $\{x | x > 2, x \in R\}$



   (b) $\{x | x \leqq 3, x \in R\}$



   (c) $\{x | x < 6, x \in R\}$



Exercise #9, page 28

1. (a) Add 4 to both sides.

   (b) Multiply each side by -1 (and change arrow).

   (c) Subtract 4 from both sides.

   (d) Divide each side by $\frac{3}{5}$. (i.e. multiply each side by $\frac{5}{3}$.)

   (e) Divide each side by -9 (and change arrow).

   (f) Add 3 to both sides.

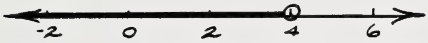
2. (a) Subtract 4x from both sides.
   Subtract 15 from both sides.
   Divide both sides by -4.
   (Change arrow.)

   (b) Apply distributive property.
   Subtract 15x from both sides.
   Subtract 7 from both sides.
   Multiply both sides by -1.
   (Change arrow.)

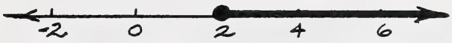
Exercise #9, continued

3. (a)
$$\begin{aligned} 15 &> 2x + 7 \\ -2x + 15 &> 7 \\ -2x &> -8 \\ x &< 4 \end{aligned}$$
Solution set is
$\{x | x < 4\}$.

(b)
$$\begin{aligned} -3x + 6 &\leqq 5x - 10 \\ -8x + 6 &\leqq -10 \\ -8x &\leqq -16 \\ x &\geqq 2 \end{aligned}$$
Solution set is
$\{x | x \geqq 2\}$.



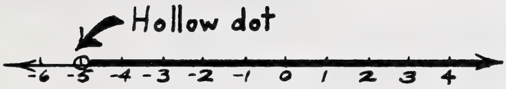
(Hollow dot indicates 4 is not included.)

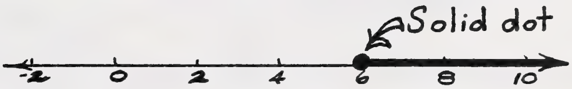


(Solid dot indicates that 2 is included.)

(c)
$$\begin{aligned} 2(1 - x) &> -3(1 + x) \\ 2 - 2x &> -3 - 3x \\ 2 + x &> -3 \\ x &> -5 \end{aligned}$$
Solution set is
$\{x | x > -5\}$.

(d)
$$\frac{x}{4} - \frac{x - 5}{6} \leqq \frac{x - 2}{3}$$
$$\overset{3}{\cancel{12}}\left(\frac{x}{\cancel{4}_1}\right) - \overset{2}{\cancel{12}}\left(\frac{x - 5}{\cancel{6}_1}\right) \leqq \overset{4}{\cancel{12}}\left(\frac{x - 2}{\cancel{3}_1}\right)$$
$$\begin{aligned} 3x - 2(x - 5) &\leqq 4(x - 2) \\ 3x - 2x + 10 &\leqq 4x - 8 \\ x + 10 &\leqq 4x - 8 \\ -3x + 10 &\leqq -8 \\ -3x &\leqq -18 \\ x &\geqq 6 \end{aligned}$$
Solution set is
$\{x | x \geqq 6\}$.





Exercise #10, page 36

1. (a)
$$\begin{aligned} x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$
Solution set is $\{4, -4\}$

(b)
$$\begin{aligned} x^2 + 5 &= 0 \\ x^2 &= -5 \\ x &= \pm\sqrt{-5} \end{aligned}$$
Solution set is ϕ .

(c)
$$\begin{aligned} 16x^2 - 1 &= 0 \\ 16x^2 &= 1 \\ x^2 &= \frac{1}{16} \\ x &= \pm\frac{1}{4} \end{aligned}$$
Solution set is $\left\{\frac{1}{4}, \frac{-1}{4}\right\}$.

(d)
$$\begin{aligned} 1 - 3x^2 &= 0 \\ -3x^2 &= -1 \\ x^2 &= \frac{1}{3} \end{aligned}$$
RATIONALIZE THIS DENOMINATOR.
$$x = \frac{\pm 1}{\sqrt{3}} = \frac{\pm 1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\pm\sqrt{3}}{3}$$
Solution set is $\left\{\frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right\}$.

Exercise #10, continued

(e)
$$-25x^2 = -1$$
$$x^2 = \frac{1}{25}$$
$$x = \pm\frac{1}{5}$$
Solution set is $\left\{\frac{1}{5}, \frac{-1}{5}\right\}$.

(f)
$$3x^2 + 3 = 2x^2 + 12$$
$$x^2 = 9$$
$$x = \pm 3$$
Solution set is $\{3, -3\}$.

Exercise #11, page 37

1. (a)
$$t^2 = 3t$$
$$t^2 - 3t = 0$$
$$t(t - 3) = 0$$
$$t = 0 \text{ or } t - 3 = 0$$
$$t = 3$$
Solution set is $\{0, 3\}$.

(b)
$$7y = y^2 - 2y$$
$$-y^2 + 9y = 0$$
$$y(-y + 9) = 0$$
$$y = 0 \text{ or } -y + 9 = 0$$
$$-y = -9$$
$$y = 9$$
Solution set is $\{0, 9\}$.

(c)
$$-3y^2 + 12y = 0$$
$$-3y(y - 4) = 0$$
$$-3y = 0 \text{ or } y - 4 = 0$$
$$y = 0 \qquad y = 4$$
Solution set is $\{0, 4\}$.

(d)
$$3x^2 + 5x = x^2 - 3x$$
$$2x^2 + 8x = 0$$
$$2x(x + 4) = 0$$
$$2x = 0 \text{ or } x + 4 = 0$$
$$x = 0 \qquad x = -4$$
Solution set is $\{0, -4\}$.

Exercise #12, page 39

1. (a)
$$(x + 7)(x - 2) = 0$$
$$x + 7 = 0 \text{ or } x - 2 = 0$$
$$x = -7 \qquad x = 2$$
Solution set is $\{-7, 2\}$.

(b)
$$(2x - 1)(3x + 7) = 0$$
$$2x - 1 = 0 \text{ or } 3x + 7 = 0$$
$$2x = 1 \qquad 3x = -7$$
$$x = \frac{1}{2} \qquad x = \frac{-7}{3}$$
Solution set is $\left\{\frac{1}{2}, \frac{-7}{3}\right\}$.

(c)
$$-(5x + 2)(x - 3) = 0$$
$$5x + 2 = 0 \text{ or } x - 3 = 0$$
$$5x = -2 \qquad x = 3$$
$$x = \frac{-2}{5}$$
Solution set is $\left\{\frac{-2}{5}, 3\right\}$.

(d)
$$(4x + 9)(x + 6) = 0$$
$$4x + 9 = 0 \text{ or } x + 6 = 0$$
$$4x = -9 \qquad x = -6$$
$$x = \frac{-9}{4}$$
Solution set is $\left\{\frac{-9}{4}, -6\right\}$.

Exercise #12, continued

(e)
$$\begin{aligned} x^2 &= -3x + 4 \\ x^2 + 3x - 4 &= 0 \\ (x + 4)(x - 1) &= 0 \end{aligned}$$
$$x + 4 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -4 \qquad\qquad x = 1$$

Solution set is $\{-4, 1\}$.

(f)
$$\begin{aligned} 6x^2 - 17x &= -5 \\ 6x^2 - 17x + 5 &= 0 \\ (3x - 1)(2x - 5) &= 0 \end{aligned}$$
$$3x - 1 = 0 \quad \text{or} \quad 2x - 5 = 0$$
$$3x = 1 \qquad\qquad 2x = 5$$
$$x = \frac{1}{3} \qquad\qquad x = \frac{5}{2}$$

Solution set is $\left\{\frac{1}{3}, \frac{5}{2}\right\}$.

(g)
$$\begin{aligned} 2x &= -8x^2 + 3 \\ 8x^2 + 2x - 3 &= 0 \\ (4x + 3)(2x - 1) &= 0 \end{aligned}$$
$$4x + 3 = 0 \quad \text{or} \quad 2x - 1 = 0$$
$$4x = -3 \qquad\qquad 2x = 1$$
$$x = \frac{-3}{4} \qquad\qquad x = \frac{1}{2}$$

Solution set is $\left\{\frac{-3}{4}, \frac{1}{2}\right\}$.

(h)
$$\begin{aligned} 24x^2 &= 54x + 15 \\ 24x^2 - 54x - 15 &= 0 \\ 8x^2 - 18x - 5 &= 0 \\ (4x + 1)(2x - 5) &= 0 \end{aligned}$$
$$4x + 1 = 0 \quad \text{or} \quad 2x - 5 = 0$$
$$4x = -1 \qquad\qquad 2x = 5$$
$$x = \frac{-1}{4} \qquad\qquad x = \frac{5}{2}$$

Solution set is $\left\{\frac{-1}{4}, \frac{5}{2}\right\}$.

Exercise #13, page 43

1.

$y^2 + 8y = 2$	$5x + 1 = -2x^2$	$x^2 = 6x - 12$
$y^2 + 8y - 2 = 0$	$2x^2 + 5x + 1 = 0$	$x^2 - 6x + 12 = 0$
$a = 1,\ b = 8,\ c = -2$	$a = 2,\ b = 5,\ c = 1$	$a = 1,\ b = -6,\ c = 12$
$x = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(-2)}}{2(1)}$ $= \frac{-8 \pm \sqrt{72}}{2}$ ↙SEE TABLE Pg 57, ℓB. $= \frac{-8 \pm 8.485}{2}$ $= 0.243$ or -8.243	$x = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(1)}}{2(2)}$ $= \frac{-5 \pm \sqrt{17}}{4}$ $= \frac{-5 \pm 4.123}{4}$ $= -2.281$ or -0.219	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$ $= \frac{6 \pm \sqrt{-12}}{2}$
Roots are 0.243 and -8.243.	Roots are -2.281 and -0.219.	No real roots.
Solution set is $\{0.243, -8.243\}$.	Solution set is $\{-2.281, -0.219\}$.	Solution set is ϕ.

2. (a) $3x^2 + 6x - 1 = 0$

$a = 3, \; b = 6, \; c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{(6)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{-6 \pm \sqrt{48}}{6}$$

SEE TABLE ON P. 57, LB

$$= \frac{-6 \pm 6.928}{6}$$

$$= -2.155 \text{ or } 0.155$$

(b) $3x^2 = 5(x - 1)^2$

$3x^2 = 5(x^2 - 2x + 1)$

$3x^2 = 5x^2 - 10x + 5$

$-2x^2 + 10x - 5 = 0$

$a = -2, \; b = 10, \; c = -5$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(-2)(-5)}}{2(-2)}$$

$$= \frac{-10 \pm \sqrt{60}}{-4}$$

$$= \frac{-10 \pm 7.746}{-4}$$

$$= 0.564 \text{ or } 4.437$$

Exercise #14, page 45

1. (a) $3y^2 = 25$

$$y^2 = \frac{25}{3}$$

$$y = \frac{\pm 5}{\sqrt{3}}$$

RATIONALIZE THIS DENOMINATOR.

$$y = \frac{\pm 5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$y = \frac{\pm 5\sqrt{3}}{3}$$

(b) $x^2 = 13x - 36$

$x^2 - 13x + 36 = 0$

$(x - 4)(x - 9) = 0$

$x - 4 = 0$ or $x - 9 = 0$

$x = 4 \qquad x = 9$

(c) $3x = 5x^2$

$-5x^2 + 3x = 0$

$x(-5x + 3) = 0$

$x = 0$ or $-5x + 3 = 0$

$-5x = -3$

$$x = \frac{3}{5}$$

(d) $4x^2 + 5 = 9$

$4x^2 = 4$

$x^2 = 1$

$x = \pm 1$

(e) $2x^2 - 6x - 1 = 0$

$a = 2, \; b = -6, \; c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{6 \pm \sqrt{44}}{4}$$

$$= \frac{6 \pm 6.633}{4}$$

$$= 3.158 \text{ or } -0.158$$

(f) $3x^2 - 10x + 2 = 0$

$a = 3, \; b = -10, \; c = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(2)}}{2(3)}$$

$$= \frac{10 \pm \sqrt{76}}{6}$$

$$= \frac{10 \pm 8.718}{6}$$

$$= 3.120 \text{ or } 0.214$$

Exercise #15, page 47

1. (a) $a = 4,\ b = -1,\ c = -5$

$$\begin{aligned} & b^2 - 4ac \\ &= (-1)^2 - 4(4)(-5) \\ &= 1 + 80 \\ &= 81 \end{aligned}$$

Roots are real and distinct.

(b) $a = 9,\ b = 12,\ c = 4$

$$\begin{aligned} & b^2 - 4ac \\ &= (12)^2 - 4(9)(4) \\ &= 144 - 144 \\ &= 0 \end{aligned}$$

Roots are real and equal.

(c) $z^2 + z + 1 = 0$

$a = 1,\ b = 1,\ c = 1$

$$\begin{aligned} & b^2 - 4ac \\ &= (1)^2 - 4(1)(1) \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

Roots are not real.

2. Roots are real and distinct when the discriminant is positive.

$$\left.\begin{array}{l} kx^2 - 2x + 1 = 0 \\ \text{In this case,} \\ a = k,\ b = -2,\ c = 1 \end{array}\right\} \qquad \begin{aligned} b^2 - 4ac &> 0 \\ (-2)^2 - 4(k)(1) &> 0 \\ 4 - 4k &> 0 \\ -4k &> -4 \\ k &< 1 \end{aligned}$$

The equation has two real and distinct roots when $k < 1$.

End of Lesson F

Lesson G

LINEAR EQUATIONS IN TWO VARIABLES, RELATIONS, FUNCTIONS, LINEAR FUNCTIONS

Topic One: Linear Equations in Two Variables

A. Finding Ordered Pairs That Satisfy Linear Equations in Two Variables

In Lesson F, you had some practice solving linear equations in one variable. Equations of this type involve only one unknown and the variable which represents this unknown appears to the first power only. When solving such equations, you must look for the one value of the variable which will make the equation true.

EXAMPLE: Solve the equation $2x - 4 = 9$. Then graph the solution set.

Solution

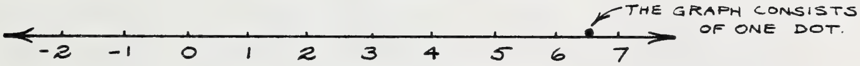
$$2x - 4 = 9$$

$$2x = 13$$

$$x = \frac{13}{2}$$

$$x = 6\frac{1}{2}$$

The solution set is $\left\{6\frac{1}{2}\right\}$.



Graph of $\{x | 2x - 4 = 9\}$

We shall now look at linear equations in two variables. Equations of this type involve two unknowns and the variables which represent these unknowns appear to the first power only. The following are examples of linear equations in two variables.

$$\left.\begin{aligned} x + y &= 6 \\ y - x &= 5 \\ 2x + 3y &= 7 \\ 3x &= 5 - 2y \end{aligned}\right\}$$

Each equation has two variables. Each variable appears to the first power.

Each equation above becomes a true statement for certain pairs of values of x and y. For each equation, we can find an infinite number of pairs that make the equation true.

EXAMPLE: Find some pairs of values of x and y that make the equation $x + y = 6$ true.

Solution The equation $x + y = 6$ is satisfied by any pair of values of x and y that add up to 6. For example, $x + y = 6$ is true when:

$x = 1$ and $y = 5$ (since $1 + 5 = 6$)

$x = \frac{1}{2}$ and $y = 5\frac{1}{2}$ $\left(\text{since } \frac{1}{2} + 5\frac{1}{2} = 6\right)$

$x = -3$ and $y = 9$ (since $-3 + 9 = 6$)

$x = 3$ and $y = 3$ (since $3 + 3 = 6$)

$x = 2$ and $y =$ _____ (since $2 +$ _____ $= 6$)

$x = -1$ and $y =$ _____ (since $-1 +$ _____ $= 6$)

$x = 0$ and $y =$ _____ (since $0 +$ _____ $= 6$)

Fill in the blanks.

In mathematics, we often write pairs of values of x and y as follows: (x, y). When you see a pair of values written like this, you know that the first number is a value of x and the second is a value of y. We call (x, y) an ordered pair of numbers.

In the example above, we could write the pairs of values that satisfy the equation $x + y = 6$ as ordered pairs. For example, rather than saying that the equation is satisfied when $x = 1$ and $y = 5$, we could say that the equation is satisfied by the ordered pair $(1, 5)$. Complete the six ordered pairs below that also satisfy the equation $x + y = 6$.

$(\frac{1}{2}, 5\frac{1}{2})$, (-3, ___), (3, ___), (2, ___) (-1, ___), (0, ___)

See example above.

Now, find six ordered pairs that satisfy the equation $x - y = 6$. (In this case, when you subtract the second number from the first, you must get a difference of 6.)

(8, 2), (12, ___), (7, ___), (15, ___), (10, ___), (6, ___)

(SINCE 8-2=6)

Self-correcting Exercise #1

Answers may be found on page 60 of this lesson.

1. Write "yes" if the ordered pair satisfies the equation and "no" if it doesn't.

Yes or No?

(a) $x + y = 9;\ (\overset{\text{VALUE OF } x}{5}, \overset{\text{VALUE OF } y}{4})$ ________

(b) $x - y = 6;\ (2, 8)$ ________

(c) $2x + y = 4;\ (2, 0)$ ________

(d) $2x - y = 1;\ (1, 0)$ ________

(e) $x + y = -5;\ (2, 3)$ ________

(f) $x + y = -5;\ (-2, -3)$ ________

(g) $2x + 3y = 7;\ (2, 1)$ ________

(h) $2x + 3y = 7;\ (-1, 3)$ ________

(i) $2x + 3y = 7;\ (1, 2)$ ________

(j) $2x + 3y = 7;\ (0, \frac{7}{3})$ ________

(k) $2x + 3y = 7;\ (\frac{7}{2}, 0)$ ________

(l) $x - 4y = 1;\ (7, -2)$ ________

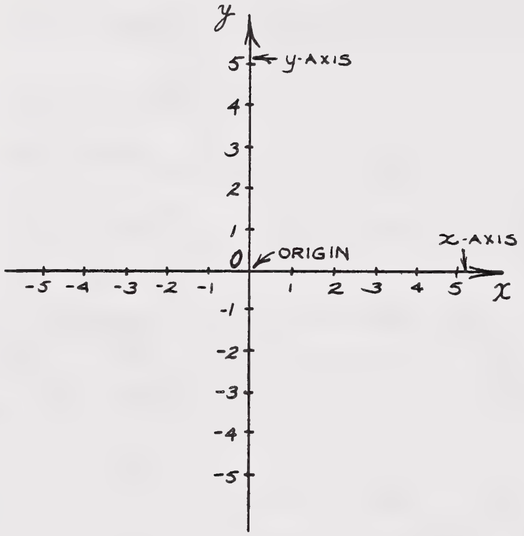
B. The Coordinate Plane

The solution to a linear equation in one variable is a single real number and can be graphed on the real number line. (See page 1 of this lesson.)

When an equation involves two variables, its solution set consists of ordered pairs of numbers. Since these pairs of numbers cannot be graphed on a single number line, we must devise a new scheme for graphing them.

I. Graphing Ordered Pairs

We can graph ordered pairs of numbers by drawing two number lines that intersect at right angles. The horizontal number line is called the x-AXIS and the vertical line is called the y-AXIS. The point of intersection of the two axes is called the ORIGIN and is labeled with the capital letter O. The plane in which the two axes are drawn is called the COORDINATE PLANE.



The scale for the x-axis is placed below the x-axis.

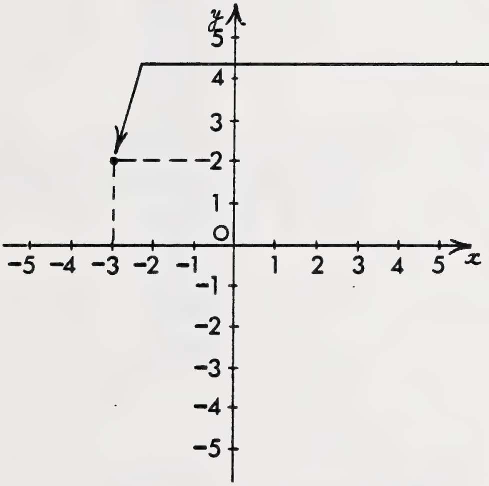
The scale for the y-axis is placed to the left of the y-axis.

There is a one-to-one correspondence between the set of all possible pairs of real numbers and the set of points in the coordinate plane. This means that:

1. Every ordered pair can be represented by a point in the plane.

2. Every point in the plane has an ordered pair associated with it.

Suppose we wished to plot the ordered pair (-3, 2) in the coordinate plane. We could locate this point by finding the first component, -3, on the x-axis and the second component, 2, on the y-axis. The point which lies directly above -3 on the x-axis and directly to the left of 2 on the y-axis corresponds to the ordered pair (-3, 2).



Point corresponding to the ordered pair (-3, 2).

In general, to find a point corresponding to the ordered pair (x, y):

1. Find the number x on the x-axis.

2. Find the number y on the y-axis.

3. Draw an imaginary vertical line through the point x on the x-axis.

4. Drawn an imaginary horizontal line through the point y on the y-axis.

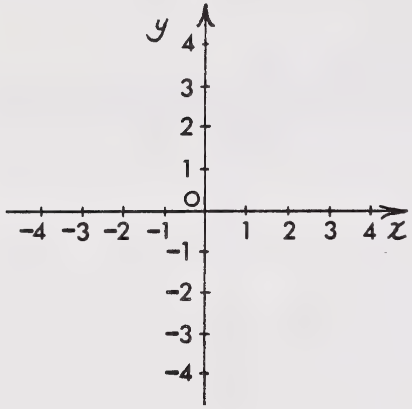
5. Mark in the point where these two lines intersect. This point is the graph of (x, y).

Self-correcting Exercise #2

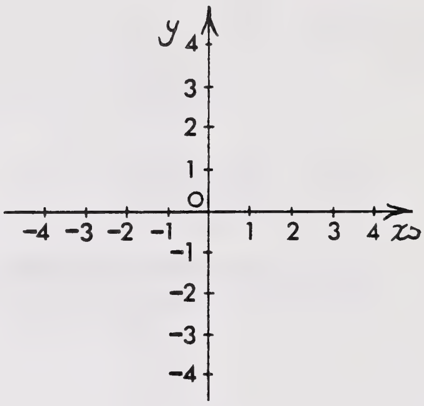
Answers may be found on page 60 of this lesson.

1. Locate each given point in the coordinate plane provided.

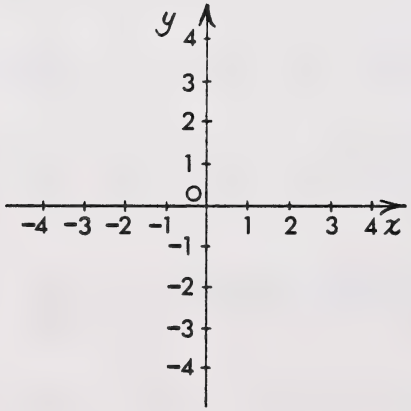
(a) (-2, -4)


y
4
3
2
1
O
-4 -3 -2 -1
1 2 3 4 x
-1
-2
-3
-4

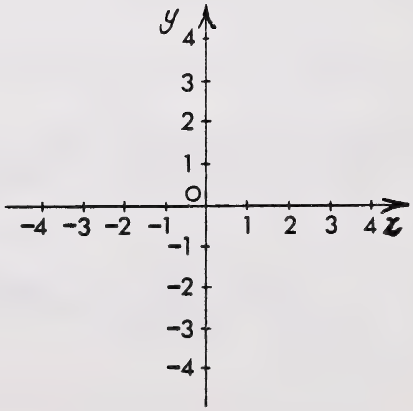
(b) (3, 2)


y
4
3
2
1
O
-4 -3 -2 -1
1 2 3 4 x
-1
-2
-3
-4

(c) (3, -2)

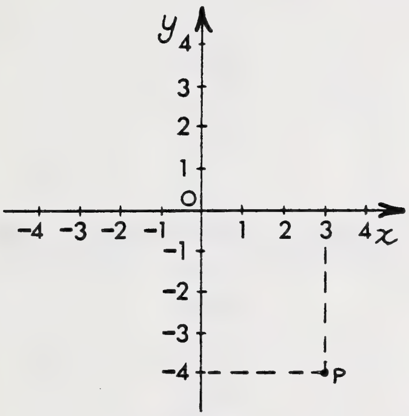

y
4
3
2
1
O
-4 -3 -2 -1
1 2 3 4 x
-1
-2
-3
-4

(d) (-2, 3)


y
4
3
2
1
O
-4 -3 -2 -1
1 2 3 4 x
-1
-2
-3
-4

II. Finding the Coordinates of a Point

How can we find the ordered pair which corresponds to point P in the coordinate plane below?



We can find the first component of the ordered pair by drawing an imaginary vertical line through P and noting that it cuts the x-axis at 3. Thus, the first component of the ordered pair is 3.

We can find the second component by drawing an imaginary horizontal line through P and noting that it cuts the y-axis at -4. Thus, the second component of the ordered pair is -4.

The ordered pair which corresponds to point P is (3, -4). 3 and -4 are called the COORDINATES of the point. The first coordinate, 3, is called the ABSCISSA of point P and the second coordinate, -4, is called the ORDINATE.
In general, to find the ordered pair (x, y) which corresponds to a given point:

1. Draw an imaginary vertical line through the point to find the value of x.

2. Draw an imaginary horizontal line through the point to find the value of y.

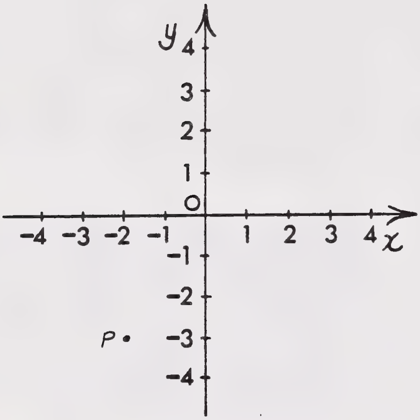
3. Write the ordered pair (x, y).

Self-correcting Exercise #3

Answers may be found on page 61 of this lesson.

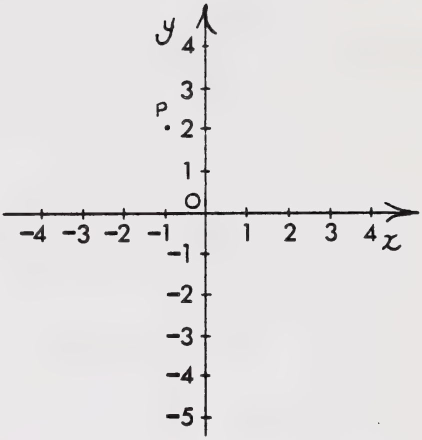
1. Find the ordered pair which corresponds to each given point.

(a)


y
4
3
2
1
O
-1
-2
-3
-4
-4 -3 -2 -1
1 2 3 4
x
P

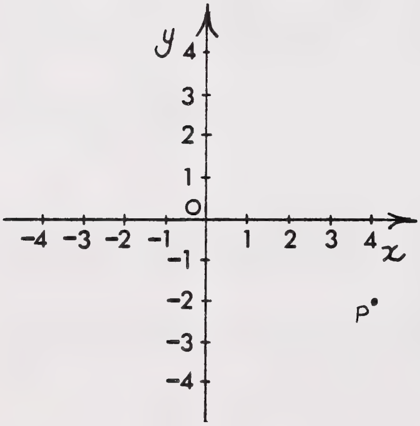
Ordered pair: __________

(b)


y
4
3
2
1
O
-1
-2
-3
-4
-5
-4 -3 -2 -1
1 2 3 4
x
P

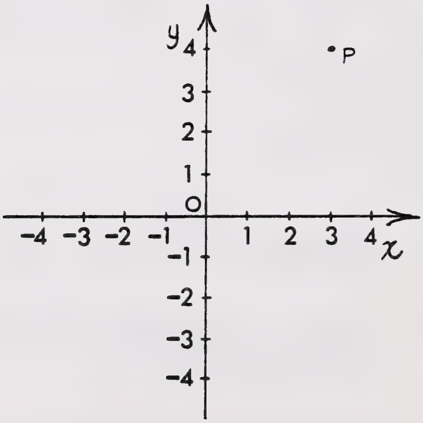
Ordered pair: __________

(c)


y
4
3
2
1
O
-1
-2
-3
-4
-4 -3 -2 -1
1 2 3 4
x
P

Ordered pair: __________

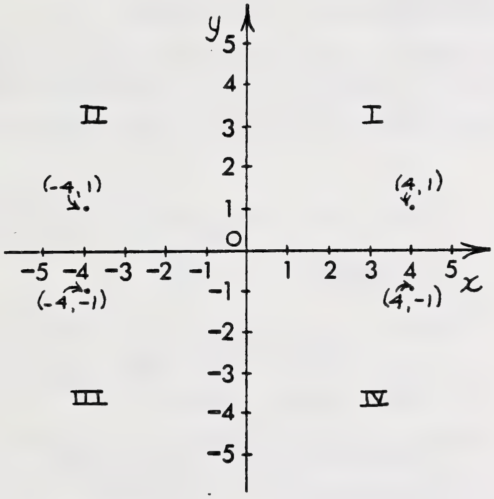
(d)


y
4
3
2
1
O
-1
-2
-3
-4
-4 -3 -2 -1
1 2 3 4
x
P

Ordered pair: __________

III. Four Quadrants of the Coordinate Plane

The x- and y-axes divide the coordinate plane into four regions called QUADRANTS. The quadrants are named using the numerals I, II, III, and IV. (See diagram below.) The signs of the coordinates of a point determine the quadrant in which the point lies.



Quadrants are numbered in a counter-clockwise direction, beginning with the top right-hand quadrant.

Point (4, 1) lies in Quadrant I.
Point (-4, 1) lies in Quadrant II.
Point (-4,-1) lies in Quadrant III.
Point (4, -1) lies in Quadrant IV.

In general:

1. All points which lie in Quadrant I have two positive coordinates. For example, the point which represents the ordered pair (4, 1) lies in Quadrant I and has the two positive coordinates 4 and 1. Give four other ordered pairs that represent points that lie in Quadrant I.

   (2,7) ____, ____, ____, ____

2. All points which lie in Quadrant II have negative first coordinates and positive second coordinates. For example, the point which represents the ordered pair (-4,1) lies in Quadrant II and has the negative first coordinate -4 and the positive second coordinate 1. Give four other ordered pairs that represent points that lie in Quadrant II.

   (-5, 6) ____, ____, ____, ____

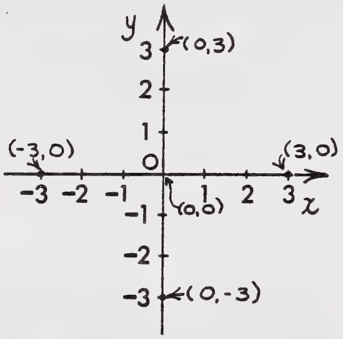
3. All points which lie in Quadrant III have two negative coordinates. For example, the point which represents the ordered pair (-4, -1) lies in Quadrant III and has the two negative coordinates -4 and -1. Give four other ordered pairs that represent points that lie in Quadrant III.

   (-3, -5) ____, ____, ____, ____

4. All points which lie in Quadrant IV have positive first coordinates and negative second coordinates. For example, the point (4, -1) lies in Quadrant IV and has the positive first coordinate 4 and the negative second coordinate -1. Give four other ordered pairs that represent points that lie in Quadrant IV.

(6,-3), __________, __________, __________

Any point which has one coordinate of zero does not lie in any of the four quadrants. It is located on either the x- or y-axis. The one point which has two zero coordinates is located at the origin (i.e. at the point of intersection of the x- and y-axes).



The points (0,3) and (0,-3) lie on the y-axis. They have first coordinates of zero.

The points (-3,0) and (3,0) lie on the x-axis. They have second coordinates of zero.

The point (0,0) lies at the intersection of the axes.

In general:

1. All points which lie on the vertical y-axis have first coordinates of zero. For example, the points which represent the ordered pairs (0,3) and (0, -3) lie on the y-axis. Give four other ordered pairs that represent points that lie on the y-axis.

(0,7), __________, __________, __________

2. All points which lie on the horizontal x-axis have second coordinates of zero. For example, the points which represent the ordered pairs (-3,0) and (3,0) lie on the x-axis. Give four other ordered pairs that represent points that lie on the x-axis.

(6,0), __________, __________, __________

Self-correcting Exercise #4

Answers may be found on page 61 of this lesson.

1. In what quadrant does the point corresponding to each ordered pair lie?

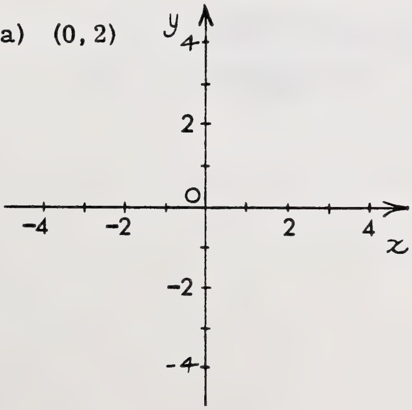
		Quadrant
(a)	(2, -5)	IV
(b)	(-5, -2)	
(c)	(3, 4)	
(d)	(-3, 2)	
(e)	(4, -4)	
(f)	(-2, -4)	

2. Decide if the point corresponding to each ordered pair lies on the x- or y-axis.

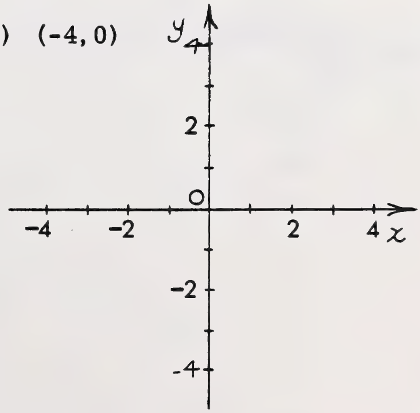
		x- or y-axis?
(a)	(4, 0)	x-axis
(b)	(-1, 0)	
(c)	(0, -5)	
(d)	(0, 2)	
(e)	(-5, 0)	
(f)	(0, 5)	

3. Locate each point in the coordinate plane provided.

(a) (0, 2)



(b) (-4, 0)



C. Graphing the Solution Set of a Linear Equation in Two Variables

We can use the coordinate plane to graph ordered pairs that satisfy a linear equation in two variables. Any such equation has an unlimited number of ordered pairs that represent solutions. Although we cannot list all these solutions, we can find a few ordered pairs that satisfy the equation, graph these ordered pairs in the coordinate plane, and then try to find some pattern in the graph.

EXAMPLE: Draw the graph of the equation $y = \frac{x}{2}$.

Solution

Substitute a few values of x into the equation $y = \frac{x}{2}$ and find the corresponding values of y.

When $x = -4$, $y = \frac{-4}{2} = -2$

When $x = -2$, $y = \frac{-2}{2} = -1$

When $x = 0$, $y = \frac{0}{2} = 0$

When $x = 2$, $y = \frac{2}{2} = 1$

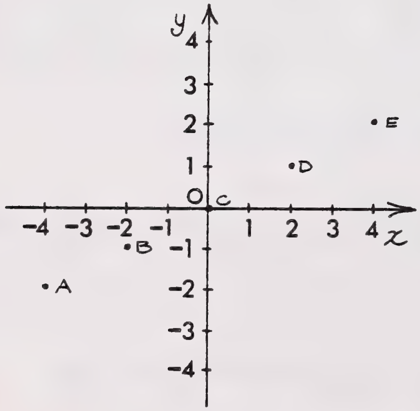
When $x = 4$, $y = \frac{4}{2} = 2$

Arrange the pairs of values of x and y in a table. Put the values of x in the first row of the table and the corresponding values of y directly below them in the second row of the table.

WHEN $x = -4$, $y = -2$. THIS PAIR OF VALUES REPRESENTS THE ORDERED PAIR $(-4, -2)$.

x	-4	-2	0	2	4
y	-2	-1	0	1	2

Locate the pairs of values on a graph.



Point A corresponds to $(-4, -2)$.

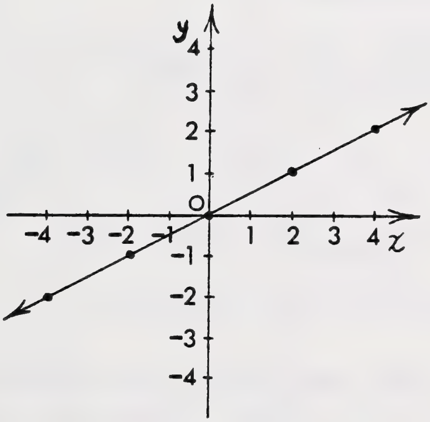
Point B corresponds to $(-2, -1)$.

Point C corresponds to $(0, 0)$.

Point D corresponds to $(2, 1)$.

Point E corresponds to $(4, 2)$.

When finding ordered pairs which satisfy the equation $y = \frac{x}{2}$, we could have substituted any real value of x in the equation and found a corresponding real value of y. Thus, for any real value of x along the x-axis, we could find a corresponding real value of y along the y-axis so that (x, y) satisfies the equation. This leads us to believe that the graph of $y = \frac{x}{2}$ is a straight line which passes through the points we have already located. We can use a ruler to draw a straight line through the five points that we located on the graph. We put arrowheads at both ends of the line to indicate that the line continues indefinitely in both directions because the number of pairs of values for x and y that satisfy the equation is infinite.



When you are graphing any linear equation in two variables, the points you plot should lie in a straight line. If they don't, you must have made a mistake in either setting up your table of values or plotting the points. You will have to go back and check your work.

The graph of any linear equation in two variables is a straight line.

When finding ordered pairs that satisfy a given linear equation in two variables, we usually plug in values for x and then find the corresponding values of y. This can be accomplished most easily if the equation is written with y isolated on the left side.

EXAMPLE: Find three ordered pairs of the form (x, y) that satisfy the equation $x - 3y = 7$. Then graph the solution set of the equation.

Solution

First, isolate y on the left side of the equation.

$$x - 3y = 7$$

Subtract x from both sides.

$$-3y = 7 - x$$

Divide both sides by -3.

$$y = \frac{7 - x}{-3}$$

Determine three ordered pairs by plugging in three different values of x and finding the corresponding values of y.

(i) When $x = -5$, $y = \frac{7 - (-5)}{-3} = \frac{7 + 5}{-3} = \frac{12}{-3} = -4$

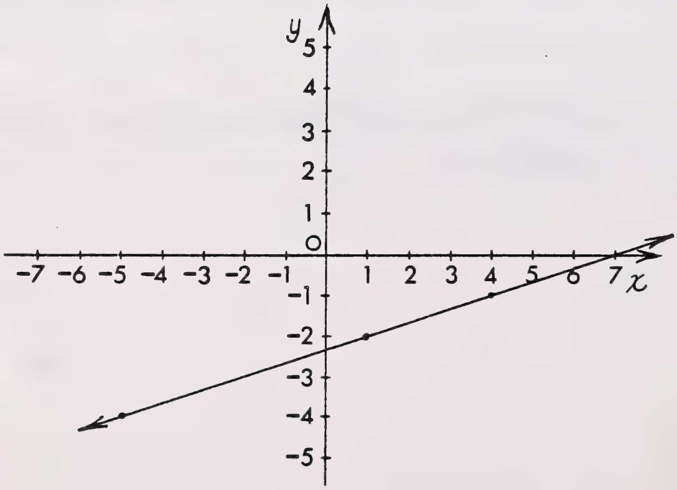
(ii) When $x = 1$, $y = \frac{7 - 1}{-3} = \frac{6}{-3} = -2$

(iii) When $x = 4$, $y = \frac{7 - 4}{-3} = \frac{3}{-3} = -1$

Thus, three ordered pairs that satisfy the equation $x - 3y = 7$ are (-5, -4), (1, -2), and (4, -1). These ordered pairs could be written in tabular form as follows:

x	-5	1	4
y	-4	-2	-1

Graph:



Self-correcting Exercise #5

Answers may be found on page 62 of this lesson.

1. Evaluate y for the given value of x.

(a) $y = -2x + 3$
When $x = 5$,
$y = (-2)(5)+3$
$= -10+3$
$= -7$

(b) $y = x + 7$
When $x = -2$,
$y =$

(c) $y = 5x$
When $x = -1$
$y =$

(d) $y = x - 4$
When $x = -3$,
$y =$

(e) $y = \frac{x + 5}{2}$
When $x = 1$,
$y =$

(f) $y = \frac{3x + 1}{2}$
When $x = -3$,
$y =$

2. Transform each equation into one in which y is isolated on the left side.

(a) $x + y = 3$

(b) $-4x + y = 7$

(c) $3x - y = 4$

(d) $2x - 2y = 6$

(e) $x - 5y = 10$

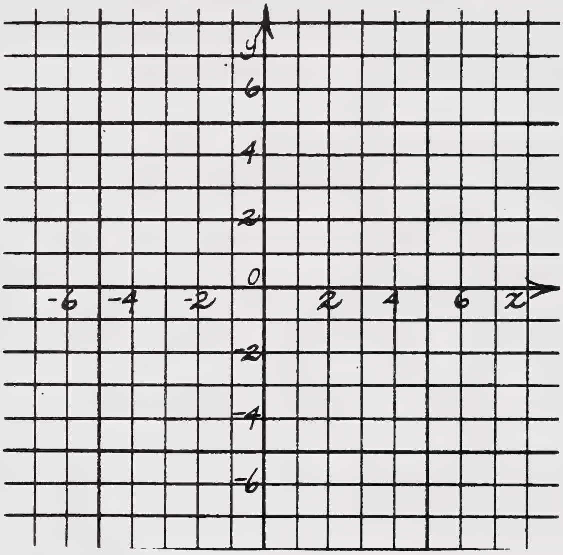
(f) $3x + 4y = 5$

3. Consider the equation $x - y = 3$.
   (a) Rewrite the equation so that y is isolated on the left side.

   (b) Fill in the following table of values.

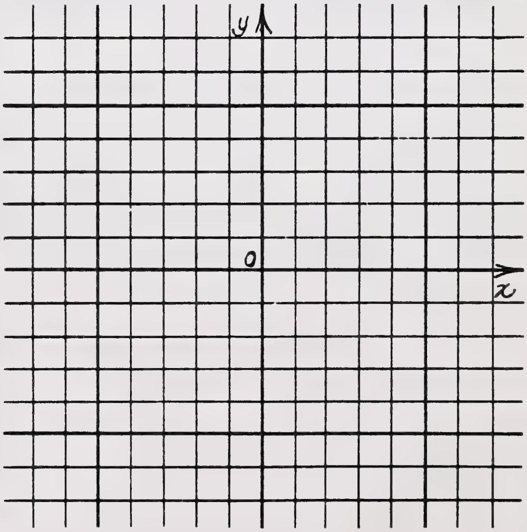
x	-4	0	2	4
y				

   (c) Graph the solution set of this equation on the coordinate axes.



4. Graph the solution set of $x + 2y = 4$.

x	-4	0	4	6
y				



EXERCISE - Linear Equations in Two Variables

1. Fill in the blanks.

(a) The solution set of a linear equation in two variables is a set of ____________ ________ of real numbers. Its graph is a ____________ ________. (See page 13.)

(b) The point of intersection of the two axes in a coordinate plane is called the ____________.

(c) The point in the coordinate plane that corresponds to (4, -3) lies in the ____________ quadrant.

(d) -3 and 2 are called the ________________ of the point corresponding to (-3, 2). -3 is called the ____________ of the point and 2 is called the ________________.

(e) The point corresponding to (0, 7) lies on the ______ - axis.

(f) All points which lie in Quadrant II have a first coordinate that is ____________ and a second coordinate that is ______________.

(g) If the ordered pair (a, -2) belongs to the solution set of the equation $2x - 3y = 10$, then $a =$ _____.

(h) There is a one-to-one correspondence between the set of ordered pairs of real numbers and the set of __________ in the ______________ plane.

2. For each equation below, list all the ordered pairs from P that satisfy it.

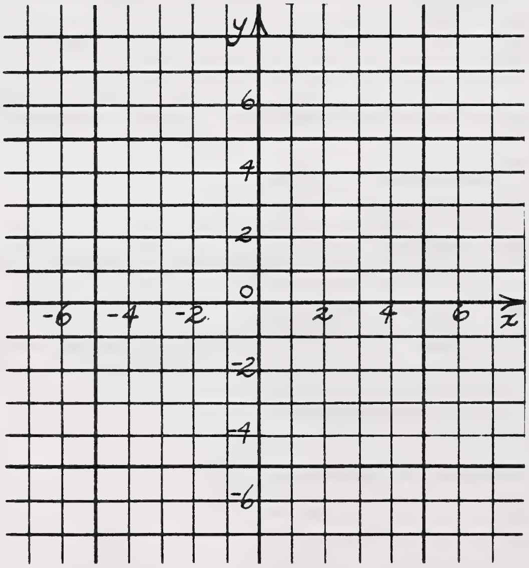
$$P = \{(1,2),\ (2,2),\ (3,6),\ (6,-2),\ (-1,-2),\ (-2,0)\}.$$

	Ordered Pairs
(a) $y = x$	(2,2)
(b) $x + y = 4$	
(c) $y = 2x$	
(d) $2x + y = -4$	
(e) $x^2 + y^2 = 5$	
(f) $y - x + 1 = 0$	

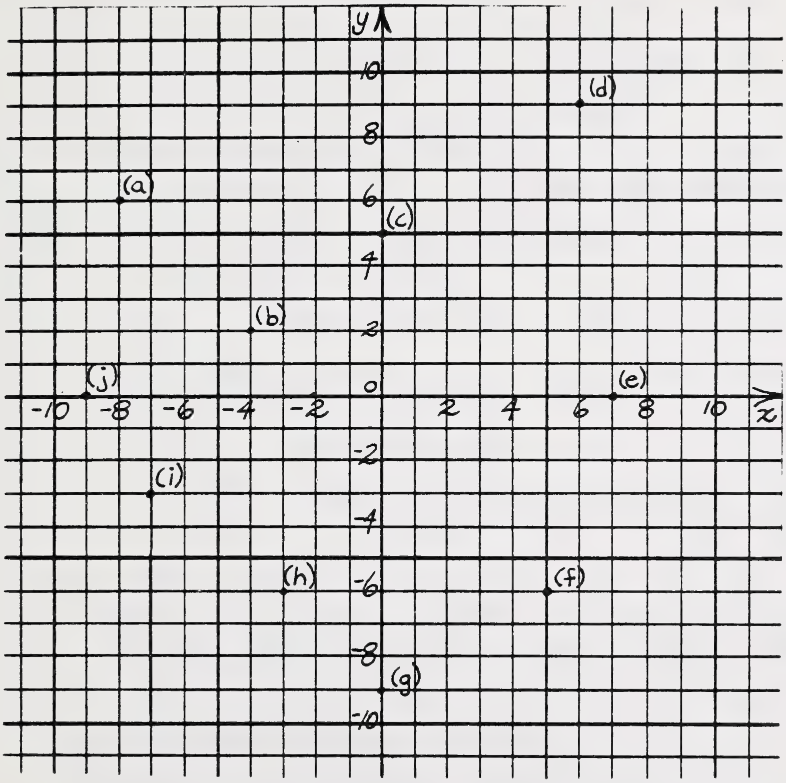
3. Graph the following set of ordered pairs in the coordinate plane below by making a dot for each ordered pair.

$$A = \{(3,4),\ (-5,6),\ (-2,-2),\ (-4,0),\ (0,3),\ (2,-4),\ (-2,-5)\}$$

(Don't write the ordered pairs on the graph.)



4. State the ordered pair that corresponds to each lettered point.


y
x
10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
2
4
6
8
10
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)

(a) (-8, 6)

(b) ______

(c) ______

(d) ______

(e) ______

(f) ______

(g) ______

(h) ______

(i) ______

(j) ______

5. Tell whether the point represented by each ordered pair below lies in Quadrant I, II, III, or IV or on the x- or y-axis.

(a) (-2, -5) Quadrant III

(b) (2, 6) ______

(c) (-5, 0) x-axis

(d) (-3, 2) ______

(e) (6, -7) ______

(f) (0, 3) ______

(g) (8, 0) ______

(h) (4, 4) ______

(i) (-4, -4) ______

6. Graph the solution set of $y - 2x - 1 = 0$.

(a) Isolate y on the left side.

$$y - 2x - 1 = 0$$

$y =$ __________

(b) Find pairs of values of x and y.

When $x = -3$, $y = 2(-3) + 1 = -6 + 1 = -5$

When $x = -1$, $y = 2(-1) + 1 =$ __________

When $x = 1$, $y =$ __________

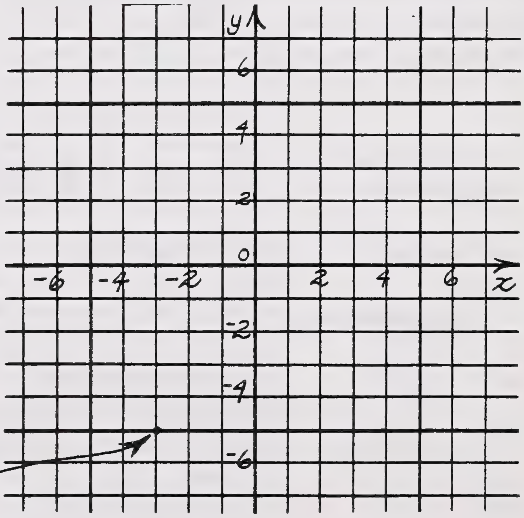
When $x = 3$, $y =$ __________

(c) Make a table of values and then plot the points. Draw a straight line through the points.

WHEN $x = -3$, $y = -5$

x	-3	-1	1	3
y	-5			

Do your plotted points all lie in the same straight line? ______
If not, look for errors in your table of values and check your plotting of the points.



THIS POINT CORRESPONDS TO (-3, -5).

7. Graph the solution set of $2y - x = 3$.

(a) Isolate y on the left side.

$$2y - x = 3$$
$$2y = \_\_\_\_\_\_\_\_$$
$$y = \_\_\_\_\_\_\_\_$$

(b) Find the value of y when x = -7, -5, 3 and 7.

(i) When x = -7,

$y = \frac{-7+3}{2} = \frac{-4}{2} = -2$

(ii) When x = -5,

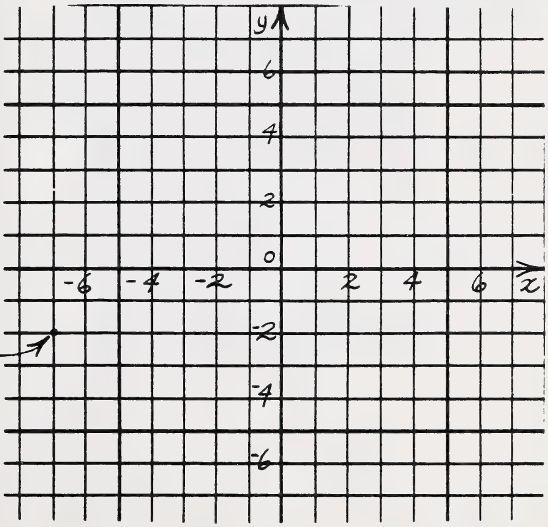
(iii) When x = 3,

(iv) When x = 7,

(c) Using the pairs of values from part (b) above, make a table of values of x and y. Then, plot the points and draw a straight line through the points. Put arrowheads at the ends of the line.

WHEN $x = -7$, $y = -2$

x	-7	-5	3	7
y	-2			



THIS POINT CORRESPONDS TO (-7, -2).

Do your plotted points all lie in the same straight line? ________ If not, look for errors in your table of values and check your plotting of the points.

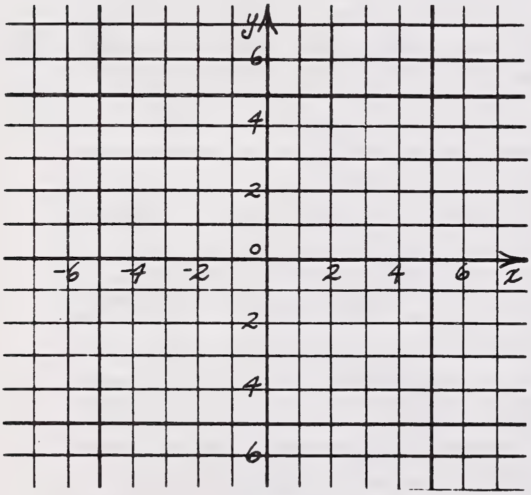
8. Transform each equation into one in which y is isolated on the left side.

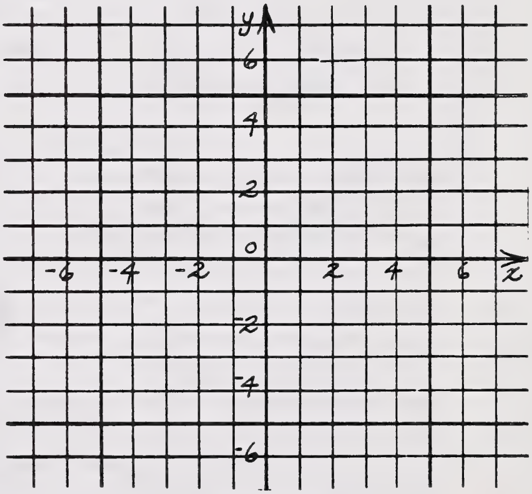
(a) $2x + y = 3$ (b) $x - y + 4 = 7$ (c) $x + 2y = 4$

(d) $x = 3 - y$ (e) $4y + 5x = -2$ (f) $-4x + 2y = 6$

9. Graph the solution set of each equation.

(a) $2x + 3y = 1$ (b) $2x - y = 0$

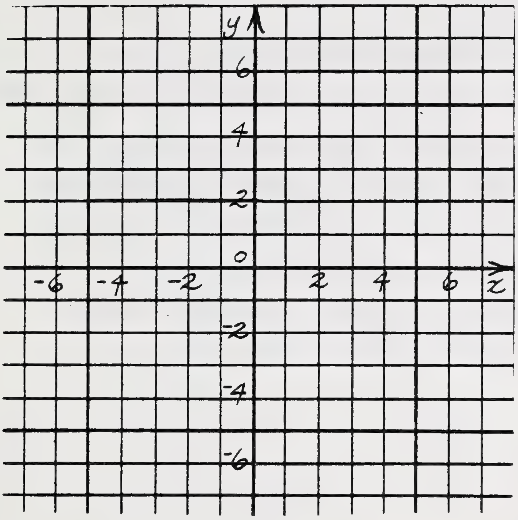

y
6
4
2
0
-6
-4
-2
2
4
6
x
2
4
6


y
6
4
2
0
-6
-4
-2
2
4
6
x
-2
-4
-6

x	
y	

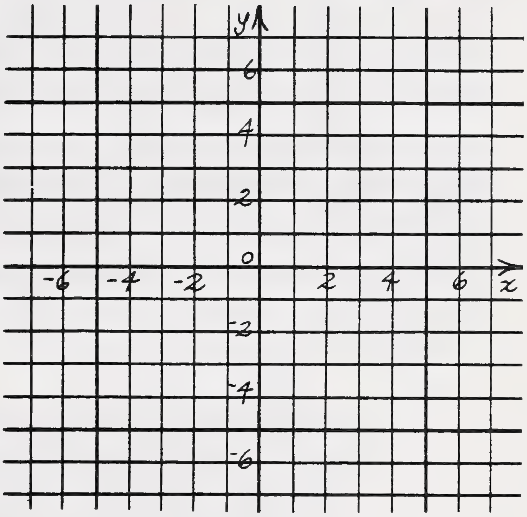
x	
y	

(c) x - y = 2


y
6
4
2
0
-6
-4
-2
2
4
6
x
-2
-4
-6

x	
y	

(d) x + y = -1


y
6
4
2
0
-6
-4
-2
2
4
6
x
-2
-4
-6

x	
y	

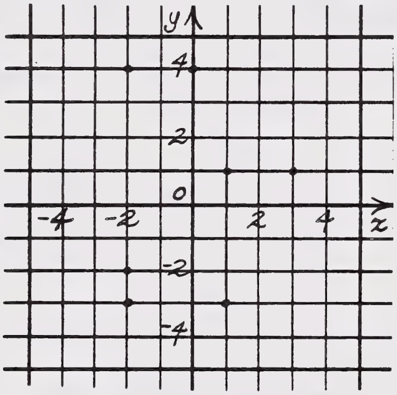
Topic Two: Relations and Functions

A. Relations

A RELATION is a set of ordered pairs of real numbers. Any relation may be graphed on the coordinate plane by finding all the points in the plane which correspond to the ordered pairs in the relation.

The set of all first components of the ordered pairs in a relation is called the DOMAIN of the relation. The set of all second components of the ordered pairs is called the RANGE of the relation.

EXAMPLE: $R_1 = \{(-2, 4), (-2, -3), (0, 4), (1, 1), (1, -3), (3, 1), (-2, -2)\}$.
R_1 is a set of ordered pairs or, in other words, a relation. This relation could be graphed on the coordinate plane as follows:



Graph of R_1

The set of all first components of the ordered pairs in R_1 is $\{-2, 0, 1, 3\}$. This set is called the domain of R_1. The set of all second components of the ordered pairs in R_1 is $\{4, -3, 1, -2\}$. This set is called the range of R_1.

State the domain and range of each of the following relations.

1. $\{(-3, 1), (-1, 1), (0, 2), (-3, 5), (4, 2)\}$ Domain: $\{-3, -1, 0, 4\}$
Range: $\{1, 2, 5\}$

2. $\{(2, -2), (2, -1), (2, 0), (2, 1), (2, 2)\}$ Domain: ____________
Range: ____________

3. $\{(1, 1), (-1, 1), (2, 4), (-2, 4), (3, 9), (-3, 9)\}$ Domain: ____________
Range: ____________

The solution set of any equation in two variables is a set of ordered pairs. Therefore, such a set must be a relation. The equation gives a rule for membership in the relation. For example, the solution set of the equation $y - x^2 = 0$ is a relation R_2 that you can specify as follows:

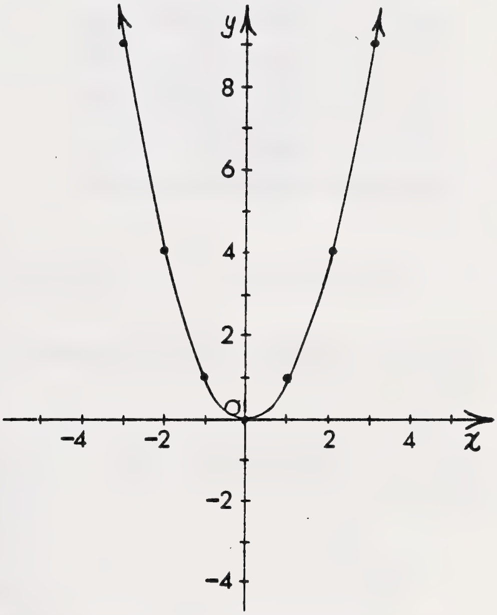
$$R_2 = \{(x, y) \mid y - x^2 = 0, \ x, y \in R\}$$

This relation is a set of ordered pairs of real numbers of the form (x, y) which satisfy the equation $y - x^2 = 0$. Since the domains of the variables are infinite sets, there are an infinite number of ordered pairs which satisfy the equation $y - x^2 = 0$. Although all the ordered pairs in R_2 cannot be tabulated, a few can be given in a table and then graphed. From the few points plotted, you can usually infer the nature of the complete graph.

$y = x^2$

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

The graph of R_2 is a curve which is called a PARABOLA.



Graph of $R_2 = \{(x, y) \mid y - x^2 = 0, \ x, y \in R\}$

In this example, the domain of R_2 is the set of real numbers, since any real number may be the first component of an ordered pair in R_2. The range of R_2 is the set of non-negative real numbers, since only ordered pairs in which the second component is zero or a positive real number appear in R_2.

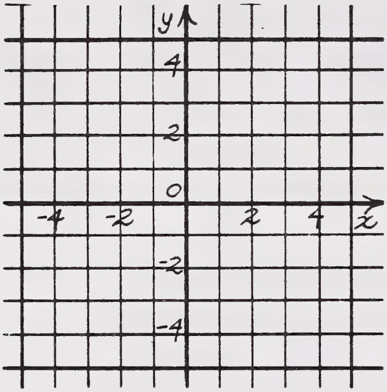
In most relations that are defined by equations, you are told that the two variables involved must be real numbers. For brevity, the notation "$x, y \in R$" may be omitted from the rule specifying the relation. Thus, in the example above, the relation R_2 could be specified as follows:

$$R_2 = \{(x, y) \mid y - x^2 = 0\}$$

Self-correcting Exercise #6

Answers may be found on page 63 of this lesson.

1. Graph the relation $R_1 = \{(-3, -2), (-1, 4), (2, 3), (-3, 3), (2, -4), (4, 0)\}$ on the coordinate axes. State the domain and range of this relation.

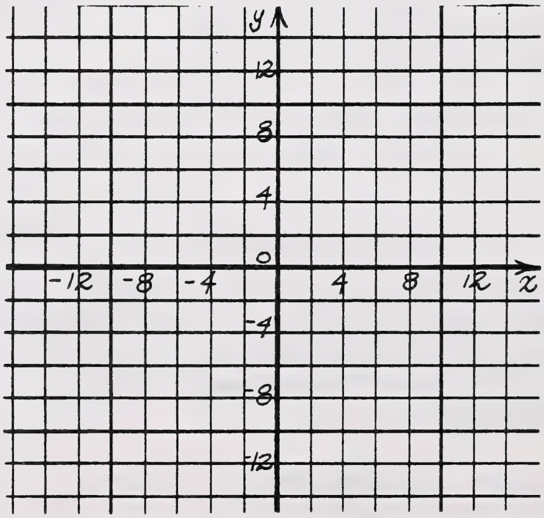

y
4
2
0
-4
-2
2
4
x
-2
-4

Domain: ____________________

Range: ____________________

2. Draw the graph of the relation $A = \{(x, y) \mid y = -3x\}$.

x	-3	-1	0	1	3
y					


y
12
8
4
0
-12
-8
-4
4
8
12
x
-4
-8
-12

3. State the domain and range of each of the following relations.

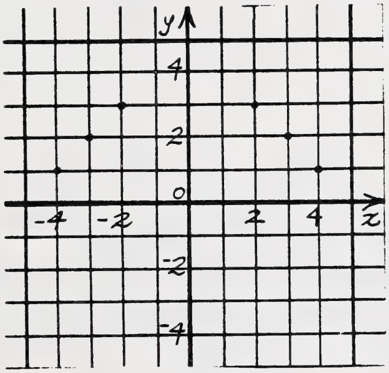
DOMAIN RANGE

(a) $\{(0,2), (1,3), (2,4)\}$ ____________ ____________

(b) $\{(-2,5), (-2,7), (3,7), (3,9)\}$ ____________ ____________

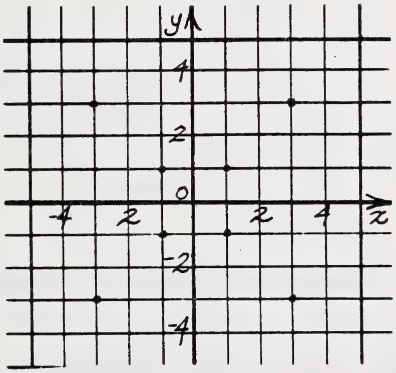
(c) $\{(-2,0), (-1,0), (0,0), (1,0)\}$ ____________ ____________

(d)



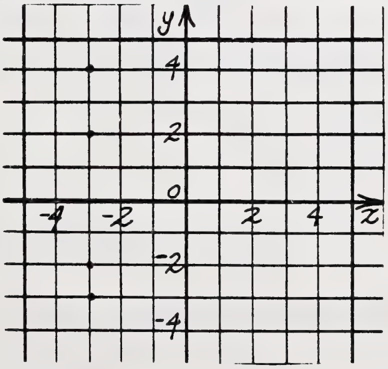
____________ ____________

(e)



____________ ____________

(f)



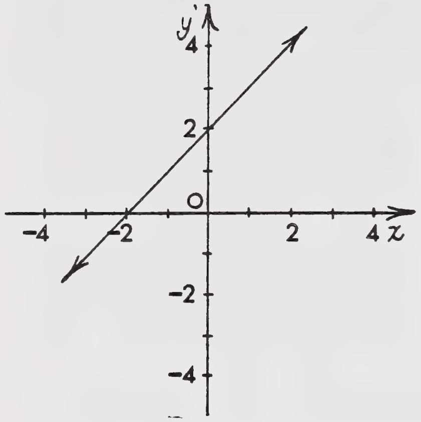
____________ ____________

B. Determining the Domain, Range and Intercepts of a Relation Graphically

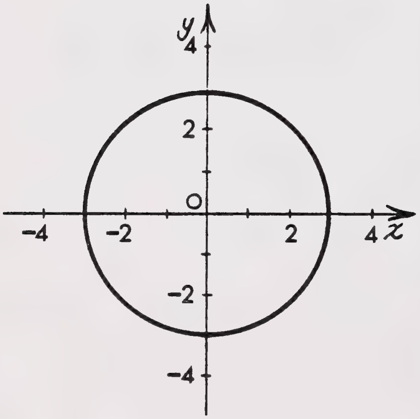
(1) Domain

The domain of a relation can be determined from its graph by noting the values of x which appear as first coordinates of points on the graph.

EXAMPLES:



Note that every value along the x-axis appears as the first coordinate of a point on the graph. Therefore, the domain of the relation is R, the set of real numbers.

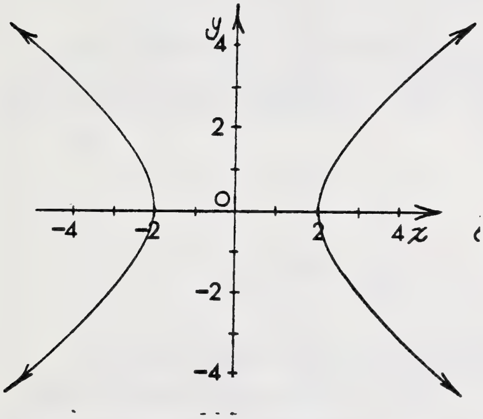


Note that only values of x which fall between -3 and 3 inclusive appear as first coordinates of points on the graph. (The graph does not lie in the region where x-coordinates are greater than 3 or less than -3.) Therefore, the domain of the relation is $\{x | -3 \leqq x \leqq 3\}$.

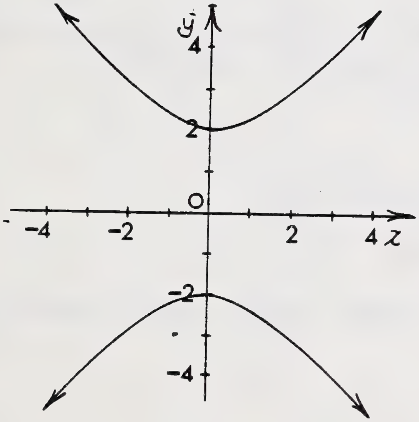
(2) Range

The range of a relation can be determined from its graph by noting the values of y which appear as second coordinates of points on the graph.

EXAMPLES:



Note that every value along the y-axis appears as the second coordinate of a point on the graph. Therefore, the range of the relation is R, the set of real numbers.



Note that only values of y that are greater than or equal to 2 or less than or equal to -2 appear as second coordinates of points on the graph. (The graph does not lie in the region where y-coordinates are between -2 and 2.) Therefore, the range of the relation is $\{y \mid y \geqq 2 \text{ or } y \leqq -2\}$.

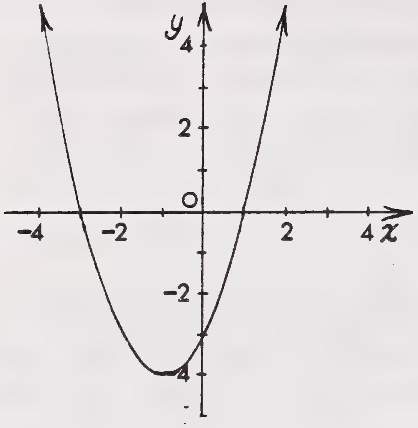
THE RANGE CONSISTS OF TWO DISTINCT SETS OF VALUES.
(i.e. IT IS NOT CONTINUOUS.)

(3) Intercepts

The x-intercepts of the graph of a relation are the abscissas (first coordinates) of the points where the graph intersects the x-axis.

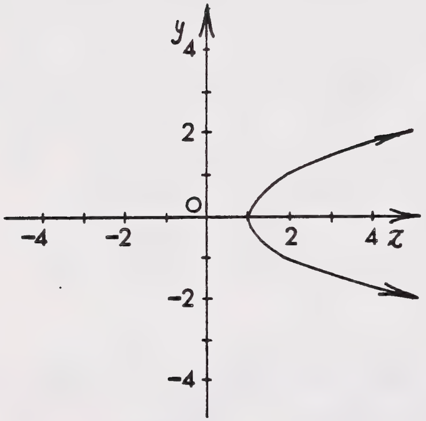
The y-intercepts of the graph of a relation are the ordinates (second coordinates) of the points where the graph intersects the y-axis.

EXAMPLES:



Note that the graph intersects the x-axis at the points whose coordinates are (-3, 0) and (1, 0). Therefore, the x-intercepts are -3 and 1.

Note that the graph intersects the y-axis at the point whose coordinates are (0, -3). Therefore, the y-intercept is -3.



Note that the graph intersects the x-axis at the point whose coordinates are (1, 0). Therefore, the x-intercept is 1.

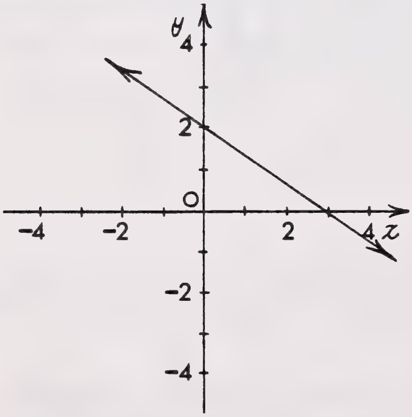
Note that the graph never intersects the y-axis. Therefore, there is no y-intercept.

Self-correcting Exercise #7

Answers may be found on page 64 of this lesson.

1. For the graph of each relation, give the domain, range, x-intercepts and y-intercepts.

(a)



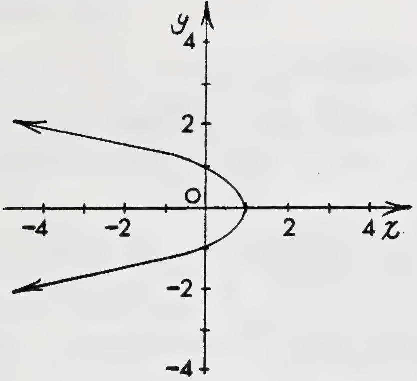
(i) Domain ____________________

(ii) Range ____________________

(iii) x-intercepts ____________________

(iv) y-intercepts ____________________

(b)



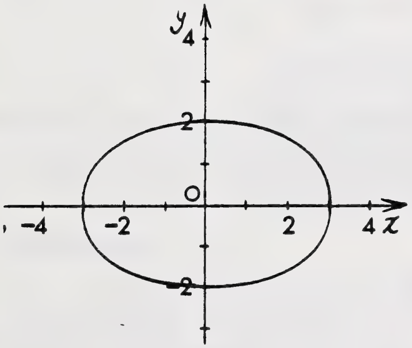
(i) Domain ______________________

(ii) Range ______________________

(iii) x-intercepts ______________________

(iv) y-intercepts ______________________

(c)



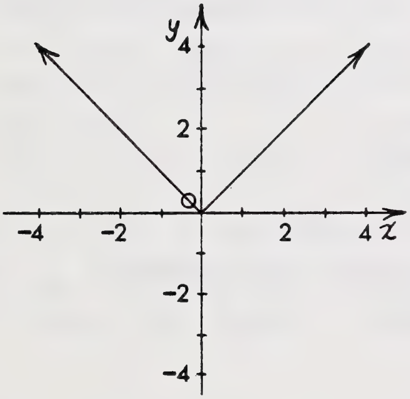
(i) Domain ______________________

(ii) Range ______________________

(iii) x-intercepts ______________________

(iv) y-intercepts ______________________

(d)



(i) Domain ______________________

(ii) Range ______________________

(iii) x-intercepts ______________________

(iv) y-intercepts ______________________

C. Determining the Domain, Range, and Intercepts of a Relation Algebraically

The domain, range and intercepts of a relation can be determined algebraically if its defining equation is known.

(1) Domain

The domain of a relation can be determined algebraically by isolating y on the left side of the defining equation and then examining the expression on the right side. The domain consists of all values of x which make the right-hand expression a real number.

EXAMPLE: $A = \{(x, y) | x + y = 3\}$

The defining equation of relation A is $x + y = 3$. Isolate y on the left side of this equation.

$$\begin{aligned} x + y &= 3 \\ y &= 3 - x \end{aligned}$$

Note that when any real number is substituted for x, the expression $3 - x$ will be a real number and hence y will be a real number. Therefore, the domain of the relation is R. (i.e. any real number can be the first component of an ordered pair in the relation.) Since the domain of the relation is R, this means that the graph will be defined for all values along the x-axis.

2. Range

The range of a relation can be determined algebraically by isolating x on the left side of the defining equation and then examining the expression on the right side. The range consists of all values of y which make the right expression a real number.

EXAMPLE: $B = \{(x, y) | y^2 - x^2 = 4\}$

The defining equation of relation B is $y^2 - x^2 = 4$. Isolate x on the left side of the equation.

$$\begin{aligned} y^2 - x^2 &= 4 \\ -x^2 &= 4 - y^2 \\ x^2 &= y^2 - 4 \\ \sqrt{x^2} &= \sqrt{y^2 - 4} \\ |x| &= \sqrt{y^2 - 4} \\ x &= \pm\sqrt{y^2 - 4} \end{aligned}$$

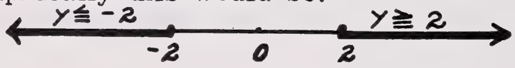
Note that when any real number is substituted for y, the expression $\sqrt{y^2 - 4}$ will not always be a real number. Only values of y which make $y^2 - 4$ greater than or equal to zero can be used. (Remember that the square root of a negative number is not a real number.)

$$\begin{aligned} \text{When } y^2 - 4 &\geqq 0, \\ y^2 &\geqq 4 \\ \sqrt{y^2} &\geqq \sqrt{4} \\ |y| &\geqq 2 \end{aligned}$$

$|y| \geqq 2$ means that y is numerically greater than or equal to 2. (i.e. y can assume values such as $2\frac{1}{4}$, $-2\frac{1}{4}$, 3, -3, $4\frac{1}{2}$ $-4\frac{1}{2}$ and so on.) Thus, the range of the relation consists of two distinct parts - values of y that are greater than or equal to 2 and values of y that are less than or equal to -2. The range is:

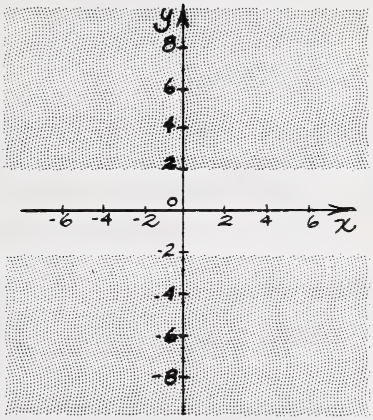
$$\{y | y \geqq 2 \text{ or } y \leqq -2\}.$$

Shown graphically this would be:



} Note the two distinct parts.

This means that any real number which is greater than or equal to 2 or less than or equal to -2 can be the second component of an ordered pair in the relation. The graph of the relation must lie in the region shaded on the coordinate plane below.



(3) Intercepts

The x-intercepts of the graph of a relation can be determined algebraically by letting $y = 0$ in the defining equation and then solving the equation for x.

The y-intercepts can be determined by letting $x = 0$ in the defining equation and then solving the equation for y.

EXAMPLE: $C = \{(x, y) \mid 4x^2 - 9y^2 = 36\}$
The defining equation of relation C is $4x^2 - 9y^2 = 36$.

To find the y-intercepts, let $x = 0$ and solve for y.

$$\begin{aligned} 4x^2 - 9y^2 &= 36 \\ \text{When } x = 0,\ 4(0)^2 - 9y^2 &= 36 \\ 0 - 9y^2 &= 36 \\ -9y^2 &= 36 \\ y^2 &= -4 \\ \sqrt{y^2} &= \sqrt{-4} \\ |y| &= \sqrt{-4} \\ y &= \pm\sqrt{-4} \quad \text{(not real)} \end{aligned}$$

Therefore, there is no y-intercept. The graph of the relation never cuts the y-axis.

To find the x-intercepts, let $y = 0$ and solve for x.

$$\begin{aligned} 4x^2 - 9y^2 &= 36 \\ \text{When } y = 0,\ 4x^2 - 9(0)^2 &= 36 \\ 4x^2 &= 36 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \\ |x| &= 3 \\ x &= \pm 3 \end{aligned}$$

Therefore, the x-intercepts are 3 and -3. The graph of the relation must cut the x-axis at the points (3, 0) and (-3, 0).

Self-correcting Exercise #8

Answers may be found on page 64 of this lesson.

1. Find the domain of the relation $A = \{(x, y) \mid x^2 + 4y^2 = 36\}$.

2. Find the range of the relation $B = \{(x, y) \mid x^2 - y^2 = 4\}$.

3. Determine the x and y-intercepts of the relation $C = \{(x, y) \mid y = x^2 - 3x - 10\}$.

4. Determine the domain, range and x-and y-intercepts of the relation $D = \{(x, y) \mid y = x^2 - 9\}$.

D. Functions

A FUNCTION is a special kind of relation in which no two distinct ordered pairs have the same first coordinate. In other words, a function is a relation in which each element in the domain corresponds to exactly one element in the range.

EXAMPLES:

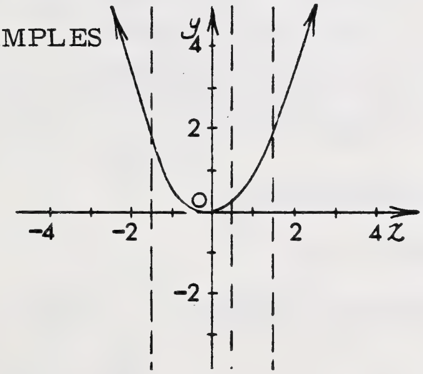
$R_1 = \{(1,2),(3,2),(5,4),(6,3)\}$ is a relation which is a function. Note that each first component appears in only one ordered pair. (i.e. each of the numbers 1, 3, 5, and 6 appear as the first component in one ordered pair.)

$R_2 = \{(2,1),(2,3),(4,5),(6,5)\}$ is a relation which is not a function. Note that the number 2 appears as a first component in the two ordered pairs (2, 1) and (2, 3).

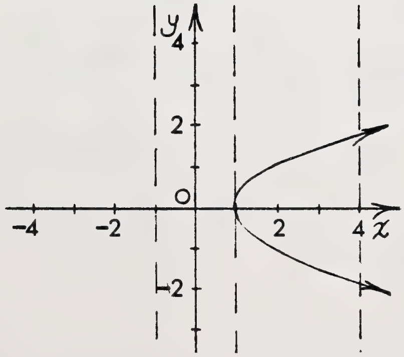
The graph of a relation also reveals whether or not the relation is a function.

If any vertical line intersects the graph of a relation in at most one point, then the relation is a function. If any vertical line intersects the graph at more than one point, the relation is not a function.

EXAMPLES



The relation which is graphed here is a function. No vertical line intersects the graph at more than one point.



The relation which is graphed here is not a function. Some vertical lines intersect the graph at two points.

We can also determine if a relation is a function by examining its defining equation. If we isolate y on the left side of the equation, we can tell if there is unique value of y for each value of x in the domain of the relation. If this is the case, then the relation is a function.

EXAMPLES:

$R_3 = \{(x, y) | x^2 + y = 3\}$ is a relation whose defining equation is $x^2 + y = 3$. Isolate y on the left side of this equation.

$$\begin{aligned} x^2 + y &= 3 \\ y &= 3 - x^2 \end{aligned}$$

Note that for each value of x in the domain of the relation, y will have the unique value $3 - x^2$. Therefore, R_3 is a function.

$R_4 = \{(x, y) | x + y^2 = 3\}$ is a relation whose defining equation is $x + y^2 = 3$. Isolate y on the left side of this equation.

$$\begin{aligned} x + y^2 &= 3 \\ y^2 &= 3 - x \\ \sqrt{y^2} &= \sqrt{3 - x} \\ |y| &= \sqrt{3 - x} \\ y &= \pm\sqrt{3 - x} \end{aligned}$$

Note that for each value of x in the domain of the relation, y can have either the value $\sqrt{3 - x}$ or $-\sqrt{3 - x}$. Since two values of y are paired with each value of x, R_4 is not a function.

In general, whenever y appears to the second degree in the defining equation of a relation, the relation is not a function.

Self-correcting Exercise #9

Answers may be found on page 65 of this lesson.

1. Decide whether or not each of the following relations is a function.

Yes or No?

(a) $A = \{(-1, 2), (-2, 4), (3, 5)\}$ __________

(b) $B = \{(4, 1), (7, 1), (8, 2), (11, 3), (9, 5)\}$ __________

(c) $C = \{(-3, 2), (-3, 5), (2, 2), (1, 2)\}$ __________

(d) $D = \{(1, 5), (3, 5), (5, 5)\}$ __________

Yes or No?

(e) $E = \{(x, y) \mid y + 3 = x\}$ ________

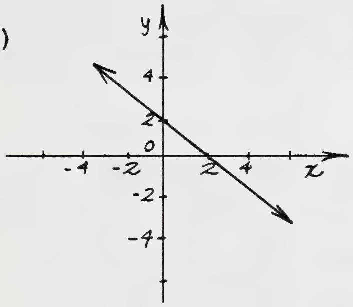
(f) $F = \{(x, y) \mid x^2 + y^2 = 36\}$ ________

(g) $G = \{(x, y) \mid y - x = x^2\}$ ________

(h) $H = \{(x, y) \mid y = 2^x\}$ ________

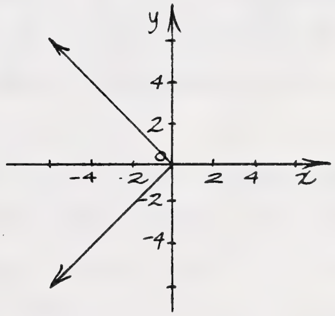
2. Decide whether or not each of the relations which is graphed below is a function.

(a)



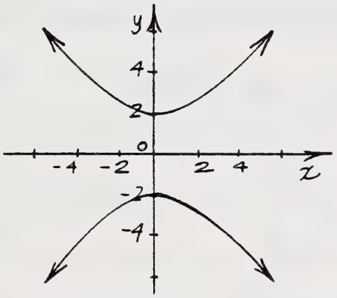
Yes or No? ________

(b)



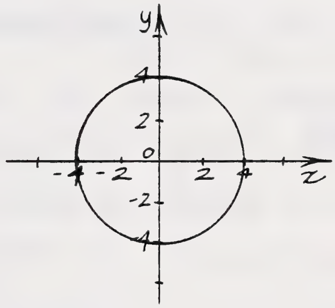
Yes or No? ________

(c)



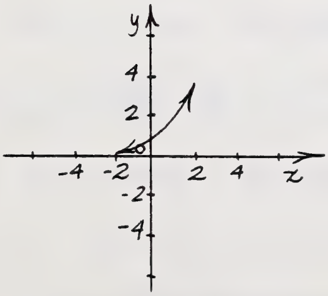
Yes or No? ________

(d)



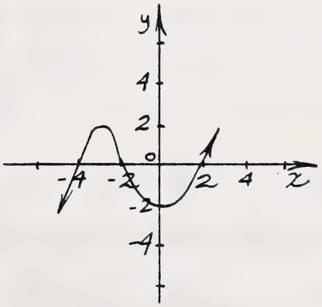
Yes or No? ________

(e)



Yes or No? ________

(f)



Yes or No? ________

E. Functional Notation

Like relations, functions can be specified using set-builder notation. Often, though, we use lower case letters to name functions rather than capitals. Also, when specifying functions, we usually isolate y on the left side of the defining equation. Then we can readily see what unique value of y is paired with each value of x.

EXAMPLE: $f = \{(x,y) \mid y = 3x^2 + 5x - 7\}$

This function is represented by the lower case letter "f". It consists of all ordered pairs of real numbers that satisfy the equation $y = 3x^2 + 5x - 7$. Note that for each value of x, y has the unique corresponding value $3x^2 + 5x - 7$. (i.e. for each value of x, there is one and only one value of y.) We can use the symbol f(x) which is read, "f at x", to denote the element in the range which is paired with each x in the domain. In this case,

$$f(x) = 3x^2 + 5x - 7.$$

We can evaluate f(x) for any real value of x. For example,

If $x = 0$, $f(0) = 3(0)^2 + 5(0) - 7 = 0 + 0 - 7 = -7$

If $x = 1$, $f(1) = 3(1)^2 + 5(1) - 7 = 3 + 5 - 7 = 1$

If $x = -1$, $f(-1) = 3(-1)^2 + 5(-1) - 7 = 3 - 5 - 7 = -9$

and so on.

If we know the defining equation of a function, we can evaluate f(x) for any real value of x.

EXAMPLE: If $f(x) = 2x^2 - 3$, evaluate f(-4), f(0) and f(7).

Solution

$f(-4) = 2(-4)^2 - 3 = 2(16) - 3 = 32 - 3 = 29$

$f(0) = 2(0)^2 - 3 = 2(0) - 3 = 0 - 3 = -3$

$f(7) = 2(7)^2 - 3 = 2(49) - 3 = 98 - 3 = 95$

Self-correcting Exercise #10

Answers may be found on page 66 of this lesson.

1. $f = \{(-3,5), (-2,6), (-1,0), (0,5), (1,6)\}$

   (a) f(-3) = ________ (b) f(0) = ________ (c) f(-1) = ________

2. $g = \{(x,y) \mid y = x^2 - 3x\}$.

   Evaluate and simplify the following.

   (a) g(2) = $(2)^2 - 3(2) = 4 - 6 = -2$

   (b) g(-1) = ________

(c) $g(\sqrt{3}) =$ $(\sqrt{3})^2 - 3(\sqrt{3}) =$ ____________________________

(d) $g(a + 1) =$ ____

(e) $g(x - 2) =$ ____

(f) $g(a^2) =$ ____

3. $f(x) = x + 1$ and $g(x) = 2x^2$.
Evaluate and simplify the following.

(a) $g[f(3)]$ (b) $f[g(x)]$

EXERCISE - Relations and Functions

1. Fill in the blanks.

(a) The first component of every ordered pair in a relation is an element of the __________________ of the relation; the second component is an element of the __________________ of the relation.

(b) A function is a special kind of __________________ in which an element of the domain can appear in only __________________ ordered pair.

(c) A vertical line may not cut the graph of a __________________ more than once.

(d) The domain of the relation $\{(x, y) \mid x^2 + y^2 = 49\}$ can be determined by isolating ______________ on the left side of the equation and determining the values of ________________ which make the expression on the right a real number.

(e) If $h(x) = -2x^2 + x$, then $h(-1) =$ __________.

(f) If $f = \{(1,2), (2,3), (3,3)\}$, then the domain of f is __________ and the range is __________.

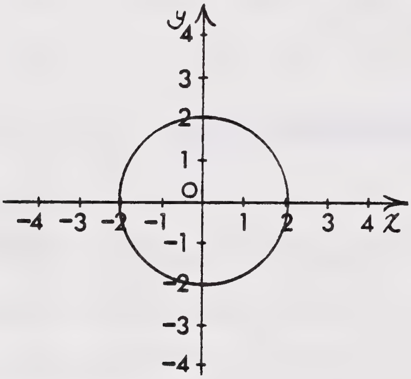
(g) If a graph cuts the x-axis at the points whose coordinates are (-3,0) and (3,0), then the x-intercepts are __________.

(h) $\sqrt{y - 3}$ is a real number when $y \geqq$ __________.

(i) The y-intercepts of a graph can be determined by letting $x =$ __________ in the defining equation and solving the resulting equation for __________.

2. For the graph of each relation, give the domain, range, x-intercepts and y-intercepts.

(a)



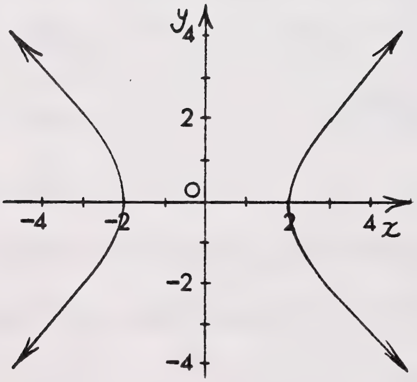
(i) Domain $\{x \mid -2 \leqq x \leqq 2\}$

(ii) Range __________

(iii) x-intercepts __________

(iv) y-intercepts __________

(b)



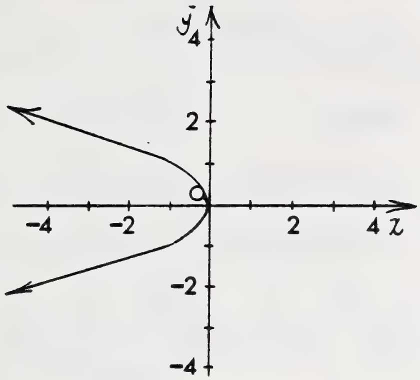
(i) Domain __________

(ii) Range __________

(iii) x-intercepts __________

(iv) y-intercepts __________

(c)



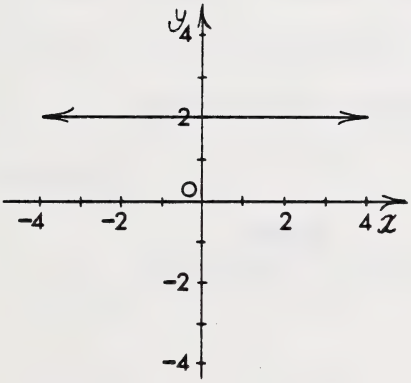
(i) Domain ________________

(ii) Range ________________

(iii) x-intercepts ________________

(iv) y-intercepts ________________

(d)



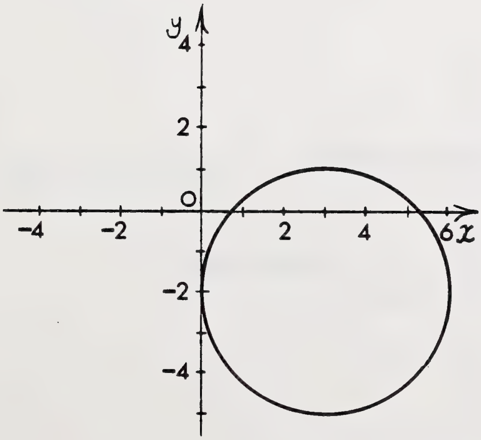
(i) Domain ________________

(ii) Range ________________

(iii) x-intercepts ________________

(iv) y-intercepts ________________

(e)



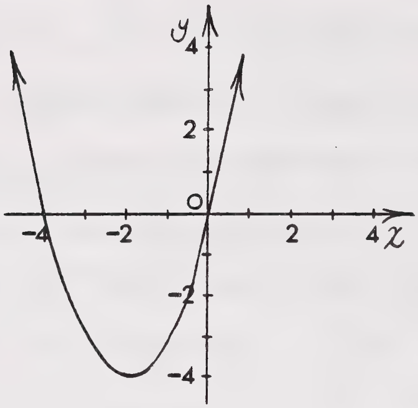
(i) Domain ________________

(ii) Range ________________

(iii) x-intercepts ________________

(iv) y-intercepts ________________

(f)



(i) Domain ____________

(ii) Range ____________

(iii) x-intercepts ____________

(iv) y-intercepts ____________

3. Which graphs in question #2, represent relations that are functions?

____

4. Find the domain and range of each relation algebraically.

(a) $\{(x, y) \mid x^2 + y^2 = 4\}$

Domain	Range
Make y the subject.	Make x the subject.

5. Find the x and y intercepts of each relation.

(a) $\{(x, y) \mid y = x^2 + 7x\}$

x-intercepts	y-intercepts
Let $y = 0$.	Let $x = 0$.

(b) $\{(x, y) \mid 9x^2 + 6y^2 = 36\}$

x-intercepts

y-intercepts

6. For the relation $\{(x, y) \mid y^2 - x^2 = 4\}$, determine the domain, range and intercepts. Then make a table of values and draw its graph.

(i) Domain

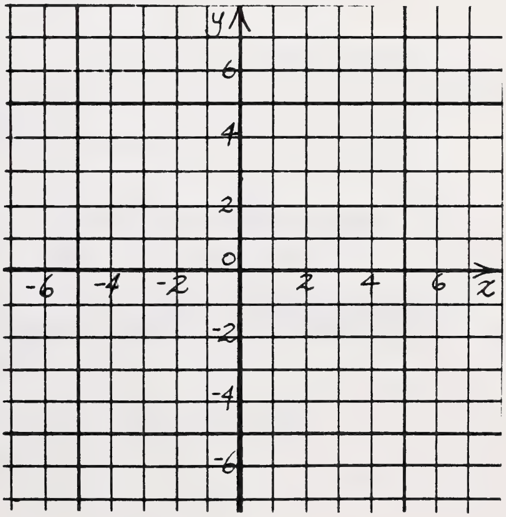
(ii) Range

(iii) x-intercepts

(iv) y-intercepts

(v) Graph

x	-6	-4	-2	0	2	4	6
y	±6.3						



7. If $f(x) = 2x^2 - x$, $x \in R$, evaluate the following expressions.

(a) $f(0) = 2(0)^2 - (0) =$ ________________

(b) $f(-1) =$ ________________

(c) $f(a) =$ ________________

(d) $f(a + 1) =$ ________________

(e) $f(y) =$ ________________

(f) $f(2) =$ ________________

8. Given:

$$t = \{(-4, 2), (-2, 1), (2, -1), (4, -2)\}$$
$$j = \{(x, y) \mid y = 3x + 1\}$$
$$f(x) = x^2 + 2$$
$$g(x) = 4 - x$$

Determine:

(a) $t(-2) =$ ________________

(b) $j(5) =$ $3(5) + 1 = 15 + 1 = 16$

(c) $f(\sqrt{3}) =$ ________________

(d) $g(a + 1) =$ ________________

(e) $f(4c) =$ ________________

(f) $\dfrac{g(d) - g(c)}{d - c} =$ ________________

EVALUATE $g(2)$ FIRST.

(g) $f[\boxed{g(2)}] =$ ________________

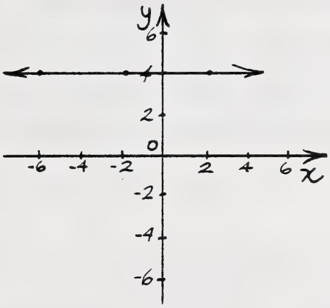
(h) $g[f(x)] =$ ________________

Topic Three: Linear Functions

A. Straight Line Graphs

A straight-line graph in the coordinate plane may be horizontal, vertical or oblique.

1. If the line is horizontal, it is the graph of a relation of the form $\{(x,y) \mid y = k\}$ where k is a real number.
EXAMPLE:



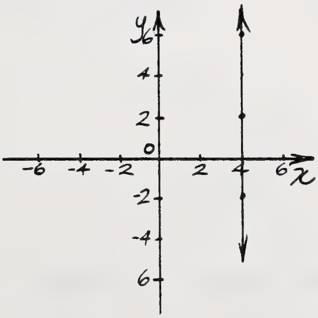
This is the graph of the relation $\{(x,y) \mid y = 4\}$. For every real value of x, y has a constant value of 4.

x	-6	-2	2
y	4	4	4

This relation is a function since for each value of x there is one and only one value of y. It is called a CONSTANT FUNCTION since the range of the function contains only one element.

> A CONSTANT FUNCTION is a function of the form $\{(x,y) \mid y = k\}$ where $k \in R$. That is, the range of the function contains only the one element k.

2. If the line is vertical, it is the graph of a relation of the form $\{(x,y) \mid x = k\}$ where k is a real number.
EXAMPLE:



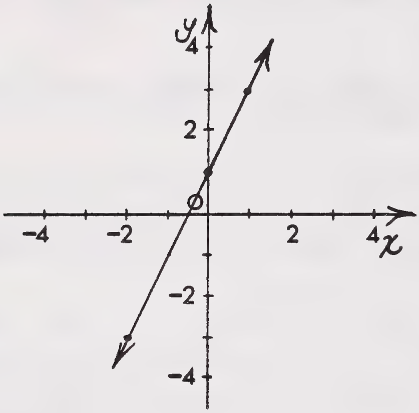
This is the graph of the relation $\{(x,y) \mid x = 4\}$. For every real value of y, x has a constant value of 4.

x	4	4	4
y	6	2	-2

This relation is not a function since an infinite number of y values are paired with the value 4 for x.

> Any relation of the form $\{(x,y) \mid x = k\}$ where $k \in R$ is not a function.

3. If the line is oblique (i.e. it is neither horizontal nor vertical), it is the graph of a relation of the form $\{(x, y) | y = mx + b)$ where $m, b \in R$ and $m \neq 0$.
EXAMPLE:



This is the graph of the relation $\{(x, y) | y = 2x + 1\}$.

x	-2	0	1
y	-3	1	3

This relation is a function since for every value of x, there is one and only one value of y. It is called a LINEAR FUNCTION because its defining equation is of the form $y = mx + b$.

A LINEAR FUNCTION is a function of the form
$$\ell = \{(x, y) | y = mx + b\}$$
where $m, b \in R$ and $m \neq 0$.

Notice that we have specified that $m \neq 0$. Should m equal 0, we would have a constant function rather than a linear function.

The defining equation of any linear function can be put in the form $y = mx + b$.
EXAMPLE: The relation $\{(x, y) | x - 2y = 3\}$ is a linear function. The defining equation can be put in the form $y = mx + b$.

$$x - 2y = 3$$
$$-2y = -x + 3$$
$$y = \frac{-x + 3}{-2}$$
$$y = \frac{-x}{-2} + \frac{3}{-2}$$
$$y = \frac{1}{2}x - \frac{3}{2}$$

The function is $\{(x, y) | y = \frac{1}{2}x - \frac{3}{2}\}$. Note that in this case

$m = \frac{1}{2}$ and $b = \frac{-3}{2}$.

Self-correcting Exercise #11

Answers may be found on page 66 of this lesson.

1. Decide whether or not each of the following relations is a linear function.

	Yes or No?
(a) $\{(x, y) \mid 2y = 8\}$	______
(b) $\{(x, y) \mid y - x = 0\}$	______
(c) $\{(x, y) \mid x + 2y = 3\}$	______
(d) $\{(x, y) \mid x + 3 = 0\}$	______
(e) $\{(x, y) \mid y = x^2\}$	______
(f) $\{(x, y) \mid x + y - 5 = 0\}$	______

2. Write each relation in the form of the general linear function and then state the values of m and b.

(a) $\{(x, y) \mid x - y = 5\}$

$x - y = 5$

$-y = -x + 5$

$y = x - 5$

$\{(x, y) \mid y = x - 5\}$

$m = 1$, $b = -5$

(b) $\{(x, y) \mid 2x - 3y = 4\}$

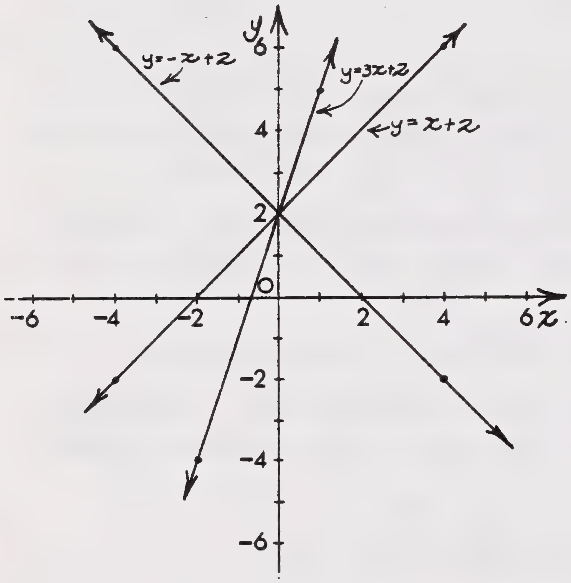
(c) $\{(x, y) \mid 3y = 4x\}$

(d) $\{(x, y) \mid 5y - 2x = 7\}$

B. The Graph of a Linear Function

The graph of any linear function in which the domain of x is R is an oblique line. This line can be located in the coordinate plane by finding three or four points whose coordinates satisfy the defining equation and then drawing a straight line through these points.

The position of the straight-line graph of any linear function depends entirely upon the values of "m" and "b" in the defining equation. Note that in the general equation $y = mx + b$, when $x = 0$, $y = b$. Therefore, for any linear function, the value of "b" tells us the y-intercept of the graph. All linear functions whose defining equations have the same value of "b" will have graphs which cut the y-axis at the same place. Study the graphs of the three linear functions $\{(x, y) | y = 3x + 2\}$, $\{(x, y) | y = x + 2\}$, and $\{(x, y) | y = -x + 2\}$, which have been drawn in the coordinate plane below.



$y = 3x + 2$

x	-2	0	1
y	-4	2	5

$y = x + 2$

x	-4	0	4
y	-2	2	6

$y = -x + 2$

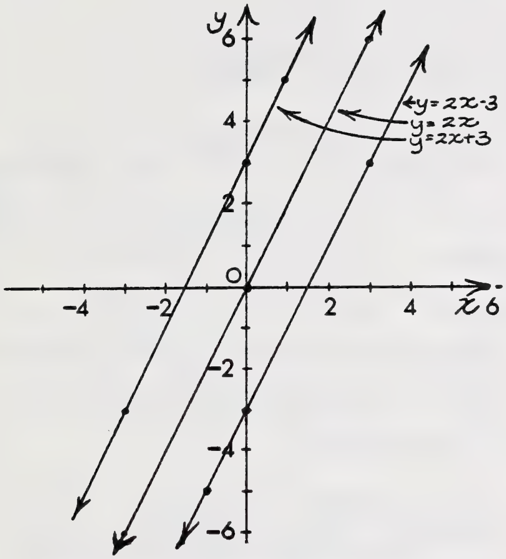
x	-4	0	4
y	6	2	-2

Note that in each defining equation, $b = 2$. The graph of each function cuts the y-axis at the point (0, 2). That is, the y-intercept of each graph is 2.

> In general, the y-intercept of the graph of the linear function $\ell = \{(x, y) | y = mx + b\}$ is "b".

The slope (or steepness) of the straight-line graph of any linear function depends entirely upon the value of m in the defining equation. If two linear functions have the same value of m then their graphs will have the same slope. This means that if their graphs are plotted in the same coordinate plane, they will be parallel.

Study the graphs of the three linear functions $\{(x, y) \mid y = 2x\}$, $\{(x, y) \mid y = 2x + 3\}$, and $\{(x, y) \mid y = 2x - 3\}$, which have been drawn in the coordinate plane below.



$y = 2x$

x	-3	0	3
y	-6	0	6

$y = 2x + 3$

x	-3	0	1
y	-3	3	5

$y = 2x - 3$

x	-1	0	3
y	-5	-3	3

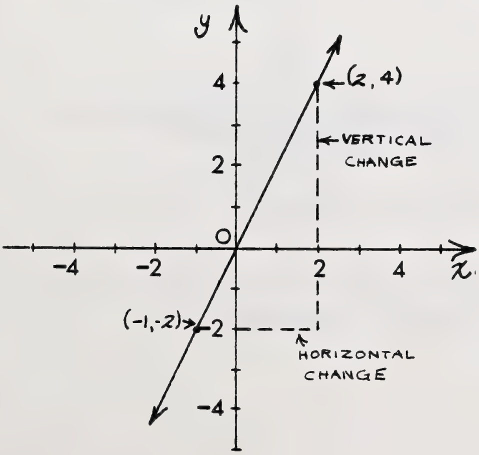
Note that in each defining equation, $m = 2$. The graphs of the three functions have the same slope and are parallel.

The slope of a straight-line graph can be determined by taking any two points on the graph and computing the following quotient:

$$\text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}}$$

(Remember that the first component of an ordered pair is called the abscissa and the second component is called the ordinate.)

For example, note how this slope formula can be applied to the graph of the function $\{(x, y) \mid y = 2x\}$.



The two points $(-1, -2)$ and $(2, 4)$ lie on the graph of the function $\{(x, y) \mid y = 2x\}$.

$$\text{Slope} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}}$$

$$= \frac{4 - (-2)}{2 - (-1)} = \frac{6}{3} = 2$$

Note that this ratio could also have been determined by subtracting the coordinates in the opposite order.

i.e. $\text{Slope} = \dfrac{(-2) - 4}{(-1) - 2} = \dfrac{-6}{-3} = 2$

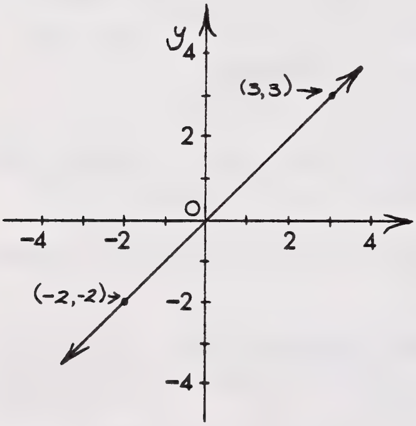
In a similar fashion, we could compute the slopes of the graphs of $\{(x, y) \mid y = 2x + 3\}$ and $\{(x, y) \mid y = 2x - 3\}$ and find them to be equal to 2. For any linear function in which $m = 2$ in the defining equation, the slope of its graph will be 2.

In general, the slope of the graph of the linear function $\ell = \{(x, y) \mid y = mx + b\}$ is "m".

$y = mx + b$ is called the SLOPE, Y-INTERCEPT FORM of the equation of a line. When the defining equation of a linear function is written in this form, we can tell at a glance what the slope and y-intercept of its straight-line graph will be. For example, if a graph is defined by $y = -3x + 5$, it will have a slope of -3 and y-intercept of 5 (since $m = -3$ and $b = 5$ in its defining equation). On the other hand, if we know the slope and y-intercept of a straight-line graph, we can write its defining equation. For example, a straight-line graph has a slope of 1 and a y-intercept of 2. This means that its defining equation must be $y = x + 2$ (since $m = 1$ and $b = 2$ in this equation).

The slope of an oblique-line graph may be positive or negative. If the line rises to the right, the slope is positive. If the line falls to the right, the slope is negative.

EXAMPLE 1:

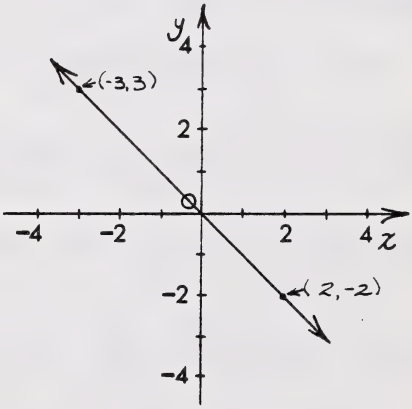


Two points on the graph are (-2, -2) and (3, 3).

$$\text{Slope} = \frac{3 - (-2)}{3 - (-2)} = \frac{5}{5} = 1$$

Graph rises to the right and the slope is positive.

EXAMPLE 2:



Two points on the graph are (-3, 3) and (2, -2).

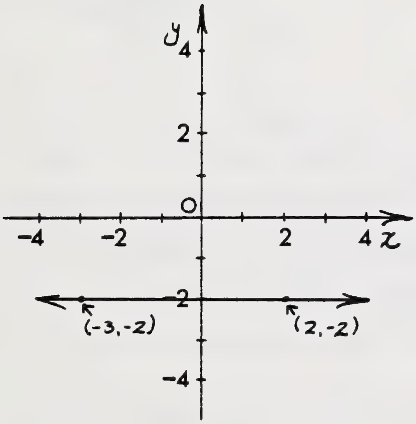
$$\text{Slope} = \frac{(-2) - 3}{2 - (-3)} = \frac{-5}{5} = -1$$

Graph falls to the right and the slope is negative.

In general, the graph of the linear function $\ell = \{(x, y) \mid y = mx + b\}$ rises to the right if m is positive and falls to the right if m is negative.

Now let us consider the slopes of horizontal and vertical lines.

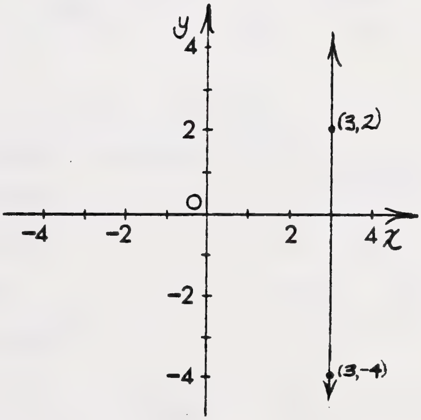
Horizontal Line



Two points on the graph are (-3, -2) and (2, -2).

$$\text{Slope} = \frac{(-2) - (-2)}{2 - (-3)} = \frac{0}{5} = 0$$

Vertical Line



Two points on the graph are (3,2) and (3, -4).

$$\text{Slope} = \frac{-4 - 2}{3 - 3} = \frac{-6}{0} = \text{undefined}$$

RECALL THAT DIVISION BY ZERO IS UNDEFINED.

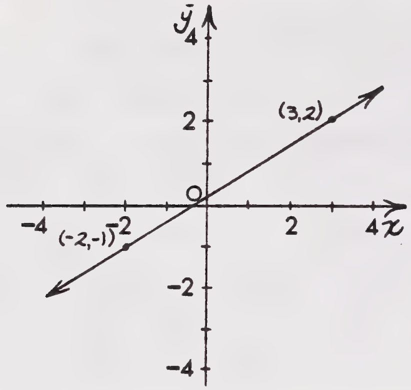
In general, the slope of any horizontal line is zero and the slope of any vertical line is undefined.

Self-correcting Exercise #12

Answers may be found on page 67 of this lesson.

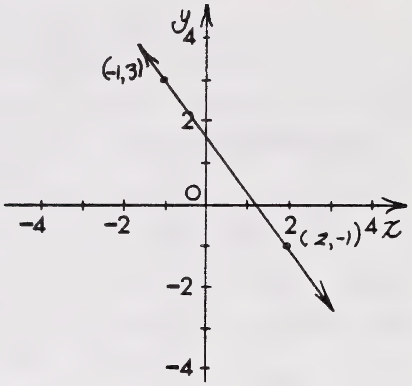
1. Find the slope of each of the following lines.

(a)


y
4
2
O
(3,2)
-4
(-2,-1)
-2
2
4 x
-2
-4

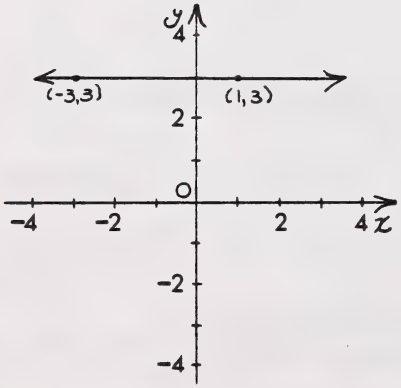
Slope =

(b)


y
4
(-1,3)
2
O
-4
-2
2
(2,-1)
4 x
-2
-4

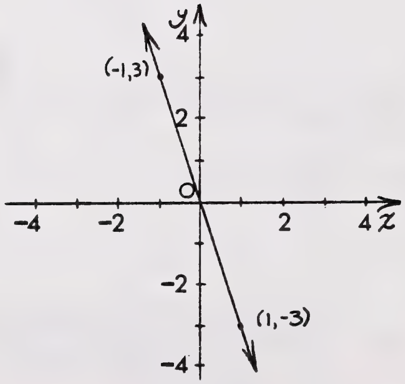
Slope =

(c)


y
4
(-3,3)
(1,3)
2
O
-4
-2
2
4 x
-2
-4

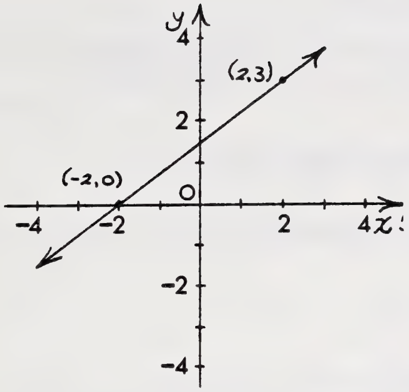
Slope =

(d)


y
4
(-1,3)
2
O
-4
-2
2
4 x
-2
(1,-3)
-4

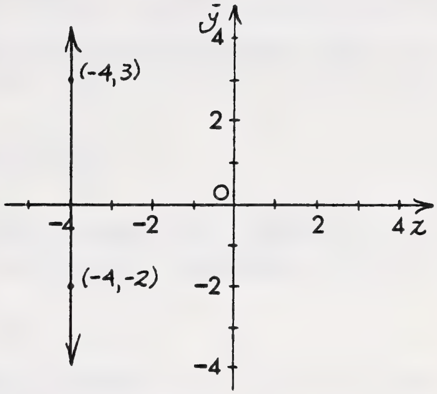
Slope =

(e)



Slope =

(f)



Slope =

2. Write the defining equation of a straight-line graph with:
   (a) a slope of 3 and y-intercept of -2. ____________
   (b) a slope of -1 and y-intercept of 6. ____________
   (c) a slope of 2 and y-intercept of 0. ____________

3. Fill in the blanks in the following chart.

Linear Equation	Form $y = mx + b$	Slope	y-intercept
(a) $2x - y = 1$	$y = 2x - 1$	2	-1
(b) $y + 3x = -5$			
(c) $y = 3 - \frac{1}{2}x$			
(d) $2x + 4y = 6$			
(e) $6x - 4y = 7$			
(f) $-2x - 3y = 6$			
(g) $5 = 2x - y$			
(h) $x - y = 1$			
(i) $y + 4x = 0$			

C. Slope-Point Form of the Equation of a Line

For any straight-line graph that has a slope of m and passes through a given point (x_1, y_1) and any other point (x, y), we know that:

$$\frac{y - y_1}{x - x_1} = m \text{ (Definition of Slope)}$$

If we multiply both sides of the above expression by $(x - x_1)$, we can write the equation in the form:

$$y - y_1 = m(x - x_1)$$

This is called the POINT-SLOPE FORM of the equation of a line. It is an algebraic statement of the geometric condition that the line must contain the point (x_1, y_1) and have a slope of m.

If we know the slope of a line and the coordinates of one point on the line, we can substitute these values into the point-slope form of the equation of a line and find its defining equation.

EXAMPLE: Write an equation of the line having slope 2 and containing (4, -3).

Solution

Since the slope is 2, $m = 2$.
Since (4, -3) lies on the line, $(x_1, y_1) = (4, -3)$

$$y - \underset{\downarrow}{y_1} = \underset{\downarrow}{m}(x - \underset{\downarrow}{x_1})$$

$$y - (-3) = 2(x - 4)$$

$$y + 3 = 2x - 8$$

$$\underline{\underline{y = 2x - 11}} \longleftarrow \text{slope, y-intercept form}$$

If we know the coordinates of two points on a line we can find the defining equation of the line.

EXAMPLE: Write an equation of the line passing through the points (-2, 4) and (5, 7).

Solution

We can use the coordinates of the two points to determine the slope of the line.

$$\text{Slope} = \frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{7 - 4}{5 - (-2)} = \frac{7 - 4}{5 + 2} = \frac{3}{7}$$

We can substitute $\frac{3}{7}$ for m and (5, 7) for (x_1, y_1) into the point-slope form of the equation of the line. [Either (5, 7) or (-2, 4) could be used for (x_1, y_1).]

$$\begin{array}{ccccccc} y & - & y_1 & = & m & (x \; - & x_1) \\ & & \downarrow & & \downarrow & & \downarrow \\ y & - & 7 & = & \frac{3}{7} & (x \; - & 5) \end{array}$$

$$y - 7 = \frac{3}{7}x - \frac{15}{7}$$

$$y = \frac{3}{7}x - \frac{15}{7} + 7$$

$$\underline{\underline{y = \frac{3}{7}x + \frac{34}{7}}}$$

Self-correcting Exercise #13

Answers may be found on page 67 of this lesson.

1. Find the defining equation of a line:

   (a) having a slope of 2 and passing through the point (-2, 5).

   (b) having a slope of -1 and passing through the point (0, 3).

(c) passing through the points (-6, -4) and (-2, -1).

EXERCISE - Linear Functions

1. Fill in the blanks.

(a) The graph of the function $\{(x, y) | y = -5\}$ is a straight line which is parallel to the ______ axis and has a slope of ______. It is called a ________________ function since there is only one value in its range.

(b) Two lines which have the same slope must be ______________.

(c) The graph of the linear function $\ell_1 = \{(x, y) | y = -x - 2\}$ has a y-intercept of ______, an x-intercept of ______, a slope of ______, and ________________ to the right.

(d) The domain of the relation $\{(x, y) | x = -4\}$ is ______ and its range is ______.

(e) The domain of the linear function $\ell_2 = \{(x, y) | y = x + 3\}$ is ________________ and its range is ___________.

(f) The defining equation of a line whose slope is $\frac{-1}{2}$, and y-intercept is 7 is ______________________.

(g) The lines whose defining equations are $y = -3x + 1$ and $y = 2x + 1$, will both cut the y-axis at the point whose coordinates are ________.

(h) The ________________ of a line is the ratio of vertical change to horizontal change.

2. Find the slopes of the lines passing through the following pairs of points.

(a) (1, -1) and (6, 3)

$$\text{Slope} = \frac{3 - (-1)}{6 - 1}$$

$$=$$

(b) (-3, 4) and (2, 4)

(c) (-2, -3) and (-1, 5)

(d) (-2, 4) and (3, -1)

(e) (2, 3) and (2, -6)

(f) (-2, -4) and (5, 6)

3. Put each equation in the form $y = mx + b$ and then state the slope and y-intercept of its graph.

(a) $y - 4x = 1$

(b) $3y = 6x - 9$

(c) $x - 3y = 1$

(d) $-2x - 3y = 5$

4. Find the defining equations of straight-line graphs with the following characteristics. (Put equations in the form $y = mx + b$.)

(a) Slope of -1 and y-intercept of 5

(b) Passes through the points $(-4,-2)$ and $(3,-4)$

(c) Slope of $\frac{5}{3}$ and passes through (-3, -4)

(d) Slope of -1 and passes through (1, -3)

(e) Slope of 4 and y-intercept of 2

(f) (3, 7) and (7, 3) are on the line.

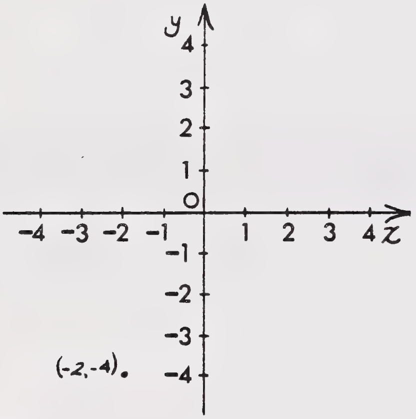
Key to Self-correcting Exercises in Lesson G

Exercise #1, page 3

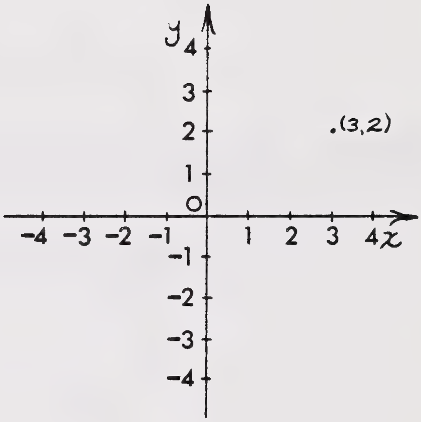
1. (a) yes; $5 + 4 = 9$
   (b) no; $2 - 8 \neq 6$
   (c) yes; $4 + 0 = 4$
   (d) no; $2 - 0 \neq 1$
   (e) no; $2 + 3 \neq -5$
   (f) yes; $(-2) + (-3) = -5$
   (g) yes; $4 + 3 = 7$
   (h) yes; $-2 + 9 = 7$
   (i) no; $2 + 6 \neq 7$
   (j) yes; $0 + 7 = 7$
   (k) yes; $7 + 0 = 7$
   (l) no; $7 + 8 \neq 1$

Exercise #2, page 6

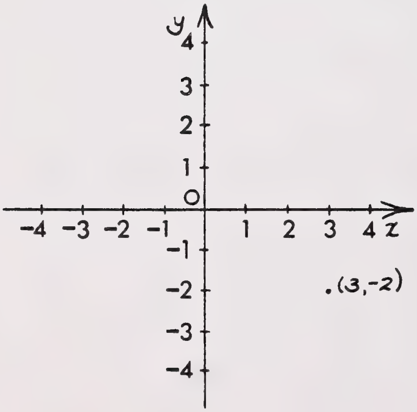
1. (a) (-2, -4)


y
4
3
2
1
0
-4 -3 -2 -1 1 2 3 4 x
-1
-2
-3
-4
(-2,-4).

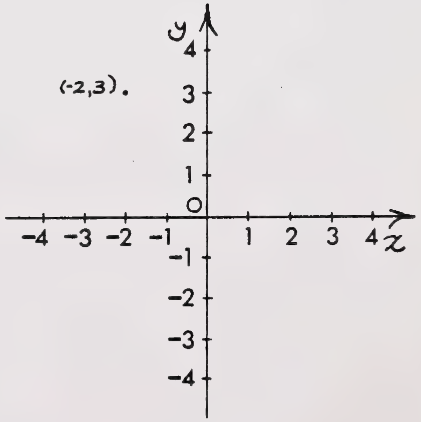
(b) (3, 2)


y
4
3
2
1
0
.(3,2)
-4 -3 -2 -1 1 2 3 4 x
-1
-2
-3
-4

(c) (3, -2)


y
4
3
2
1
0
-4 -3 -2 -1 1 2 3 4 x
-1
-2
-3
-4
.(3,-2)

(d) (-2, 3)


y
4
(-2,3).
3
2
1
0
-4 -3 -2 -1 1 2 3 4 x
-1
-2
-3
-4

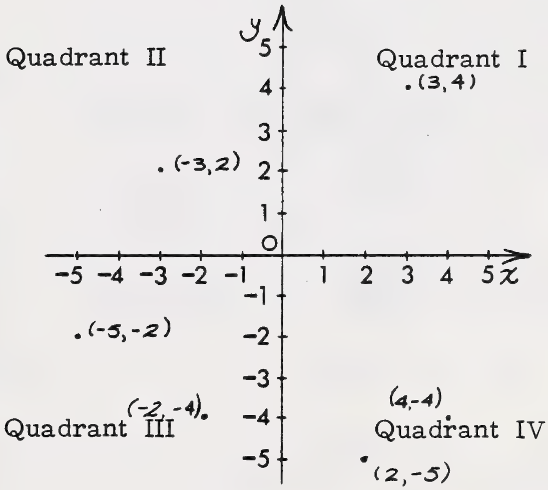
Exercise #3, page 8

1. (a) (-2, -3) (b) (-1, 2)

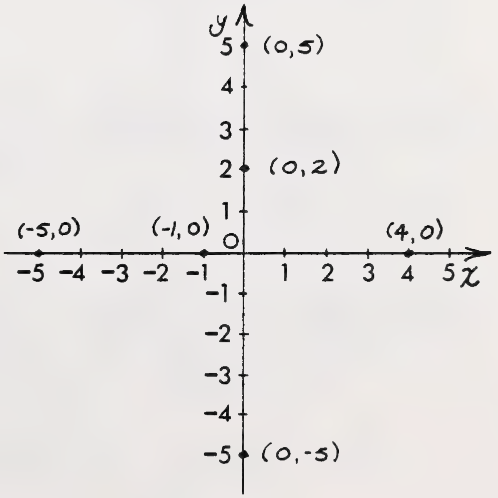
(c) (4, -2) (d) (3, 4)

Exercise #4, page 11

			Quadrant
1.	(a)	(2, -5)	IV
	(b)	(-5, -2)	III
	(c)	(3, 4)	I
	(d)	(-3, 2)	II
	(e)	(4, -4)	IV
	(f)	(-2, -4)	III

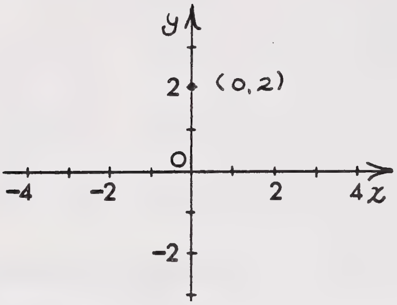

y
Quadrant II
Quadrant I
(3, 4)
(-3, 2)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5 -4 -3 -2 -1
1 2 3 4 5 x
(-5, -2)
(-2, -4)
Quadrant III
(4, -4)
Quadrant IV
(2, -5)

			x-or y-axis?
2.	(a)	(4, 0)	x-axis
	(b)	(-1, 0)	x-axis
	(c)	(0, -5)	y-axis
	(d)	(0, 2)	y-axis
	(e)	(-5, 0)	x-axis
	(f)	(0, 5)	y-axis

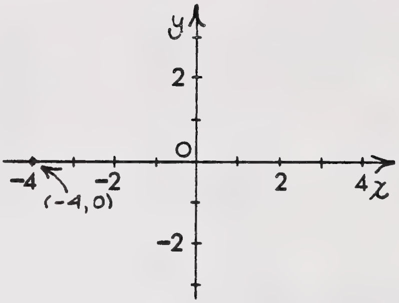

y
(0, 5)
(0, 2)
(-5, 0)
(-1, 0)
(4, 0)
0
5
4
3
2
1
-1
-2
-3
-4
-5
-5 -4 -3 -2 -1
1 2 3 4 5 x
(0, -5)

Exercise #4, continued

3. (a) (0, 2)



(b) (−4, 0)



Exercise #5, page 15

1. (a)
$$\begin{aligned} y &= -2x + 3 \\ &= (-2)(5) + 3 \\ &= -10 + 3 \\ &= \enclose{circle}{-7} \end{aligned}$$

(b)
$$\begin{aligned} y &= x + 7 \\ &= -2 + 7 \\ &= \enclose{circle}{5} \end{aligned}$$

(c)
$$\begin{aligned} y &= 5x \\ &= 5(-1) \\ &= \enclose{circle}{-5} \end{aligned}$$

(d)
$$\begin{aligned} y &= x - 4 \\ &= -3 - 4 \\ &= \enclose{circle}{-7} \end{aligned}$$

(e)
$$\begin{aligned} y &= \frac{x + 5}{2} \\ &= \frac{1 + 5}{2} \\ &= \frac{6}{2} \\ &= \enclose{circle}{3} \end{aligned}$$

(f)
$$\begin{aligned} y &= \frac{3x + 1}{2} \\ &= \frac{3(-3) + 1}{2} \\ &= \frac{-9 + 1}{2} \\ &= \frac{-8}{2} \\ &= \enclose{circle}{-4} \end{aligned}$$

2. (a)
$$\begin{aligned} x + y &= 3 \\ \underline{y = 3 - x} \end{aligned}$$

(b)
$$\begin{aligned} -4x + y &= 7 \\ \underline{y = 7 + 4x} \end{aligned}$$

(c)
$$\begin{aligned} 3x - y &= 4 \\ -y &= 4 - 3x \end{aligned}$$
Multiply every term by −1.
$$\underline{y = 3x - 4}$$

(d)
$$\begin{aligned} 2x - 2y &= 6 \\ -2y &= 6 - 2x \\ y &= \frac{6 - 2x}{-2} \end{aligned}$$
(or) $\underline{y = x - 3}$

(e)
$$\begin{aligned} x - 5y &= 10 \\ -5y &= 10 - x \\ y &= \frac{10 - x}{-5} \end{aligned}$$
(or) $\boxed{y = \frac{x - 10}{5}}$

(f)
$$\begin{aligned} 3x + 4y &= 5 \\ 4y &= 5 - 3x \end{aligned}$$
$$\boxed{y = \frac{5 - 3x}{4}}$$

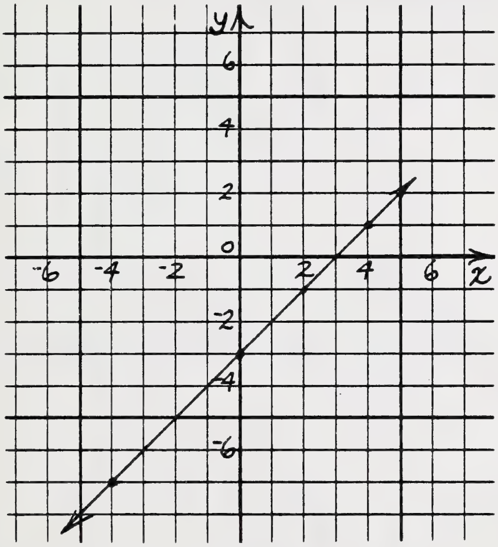
Exercise #5, continued

3. $x - y = 3$

$-y = 3 - x$

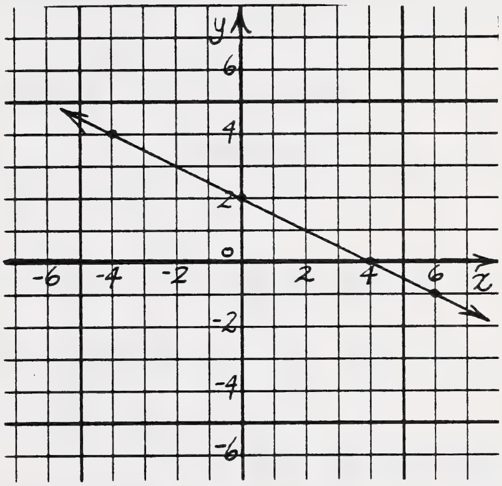
$\boxed{y = x - 3}$

x	-4	0	2	4
y	-7	-3	-1	1



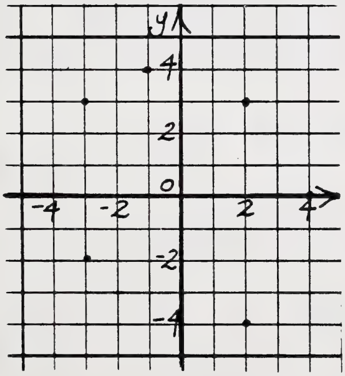
4. $\boxed{y = \frac{4 - x}{2}}$

x	-4	0	4	6
y	4	2	0	-1



Exercise #6, page 26

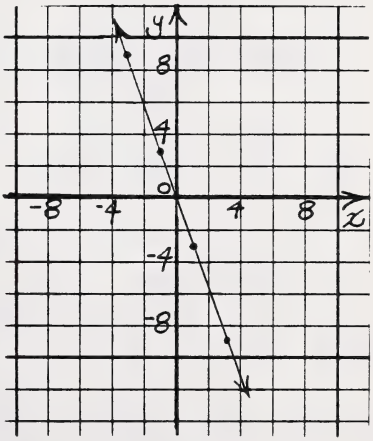
1. Graph of R_1



Domain is $\{-3, -1, 2, 4\}$.
Range is $\{-2, 4, 3, -4, 0\}$.

2. $y = -3x$

x	-3	-1	0	1	3
y	9	3	0	-3	-9



Exercise #6, continued

3. (a) Domain is $\{0,1,2\}$. Range is $\{2,3,4\}$.
   (b) Domain is $\{-2,3\}$. Range is $\{5,7,9\}$.
   (c) Domain is $\{-2,-1,0,1\}$. Range is $\{0\}$.
   (d) Domain is $\{-4,-3,-2,2,3,4\}$. Range is $\{1,2,3\}$.
   (e) Domain is $\{-3,-1,1,3\}$. Range is $\{-3,-1,1,3\}$.
   (f) Domain is $\{-3\}$. Range is $\{-3,-2,2,4\}$.

Exercise #7, page 30

1. (a)	(i) R	(ii) R	(iii) 3	(iv) 2
(b)	(i) $\{x\mid x \leqq 1\}$	(ii) R	(iii) 1	(iv) 1 and -1
(c)	(i) $\{x\mid -3 \leqq x \leqq 3\}$	(ii) $\{y\mid -2 \leqq y \leqq 2\}$	(iii) 3 and -3	(iv) 2 and -2
(d)	(i) R	(ii) $\{y\mid y \geqq 0\}$	(iii) 0	(iv) 0

Exercise #8, page 34

1.
$$\begin{aligned} x^2 + 4y^2 &= 36 \\ 4y^2 &= 36 - x^2 \\ y^2 &= \frac{36 - x^2}{4} \\ \sqrt{y^2} &= \sqrt{\frac{36 - x^2}{4}} \\ |y| &= \frac{\sqrt{36 - x^2}}{2} \\ y &= \frac{\pm\sqrt{36 - x^2}}{2} \end{aligned}$$

$\sqrt{36 - x^2}$ is real when

$$\begin{aligned} 36 - x^2 &\geqq 0 \\ -x^2 &\geqq -36 \\ x^2 &\leqq 36 \\ \sqrt{x^2} &\leqq \sqrt{36} \\ |x| &\leqq 6 \longleftarrow \text{THIS MEANS THAT } x \text{ IS NUMERICALLY LESS THAN OR EQUAL TO 6.} \end{aligned}$$

(i.e. $-6 \leqq x \leqq 6$)

Domain is $\{x\mid -6 \leqq x \leqq 6\}$

2.
$$\begin{aligned} x^2 - y^2 &= 4 \\ x^2 &= 4 + y^2 \\ \sqrt{x^2} &= \sqrt{4 + y^2} \\ |x| &= \sqrt{4 + y^2} \\ x &= \pm\sqrt{4 + y^2} \end{aligned}$$

$\sqrt{4 + y^2}$ is real when $4 + y^2 \geqq 0$. This is an unconditional inequality which is true for all real values of y.

Range is R.

Exercise #8, continued

3. $y = x^2 - 3x - 10$

If $y = 0$, $x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$

$x = 5$ or $x = -2$

The x-intercepts are 5 and -2.

$y = x^2 - 3x - 10$

If $x = 0$, $y = (0)^2 - 3(0) - 10$

$y = 0 - 0 - 10$

$y = -10$

The y-intercepts is -10.

4. Domain

$y = x^2 - 9$

For each real value of x, y will be a real number. Thus, the values of x are unrestricted. The domain is R.

Range

$y = x^2 - 9$

Make x the subject.

$x^2 = y + 9$

$x = \pm\sqrt{y + 9}$

$\sqrt{y + 9}$ is real when

$y + 9 \geqq 0$

$y \geqq -9$

The range is $\{y \mid y \geqq -9\}$.

x-intercepts

Let $y = 0$

$0 = x^2 - 9$

$x^2 = 9$

$x = \pm 3$

x-intercepts are ± 3.

y-intercepts

Let $x = 0$

$y = (0)^2 - 9$

$y = -9$

y-intercept is -9.

Exercise #9, page 36

1. (a) yes

(b) yes

(c) no; -3 appears as the first component in two ordered pairs.

(d) yes

(e) yes

(f) no; y^2 appears in the defining equation.

(g) yes

(h) yes

2. Graphs represent functions in parts (a), (e), and (f).
Graphs do not represent functions in parts (b), (c), and (d) since in each case vertical lines can cut the graphs in two places.

Exercise #10, page 38

1. (a) $f(-3) = \underline{5}$ (b) $f(0) = \underline{5}$ (c) $f(-1) = \underline{0}$

2. (a) $g(2) = (2)^2 - 3(2) = 4 - 6 = \enclose{circle}{-2}$

   (b) $g(-1) = (-1)^2 - 3(-1) = 1 + 3 = \enclose{circle}{4}$

   (c) $g(\sqrt{3}) = (\sqrt{3})^2 - 3(\sqrt{3}) = \underline{3 - 3\sqrt{3}}$

   (d) $g(a + 1) = (a + 1)^2 - 3(a + 1) = a^2 + 2a + 1 - 3a - 3 = \underline{a^2 - a - 2}$

   (e) $g(x - 2) = (x - 2)^2 - 3(x - 2) = x^2 - 4x + 4 - 3x + 6 = \underline{x^2 - 7x + 10}$

   (f) $g(a^2) = (a^2)^2 - 3(a^2) = \underline{a^4 - 3a^2}$

3. (a) $\boxed{f(x) = x + 1}$

   $\therefore f(3) = 3 + 1$

   $= \enclose{circle}{4}$

   SYMBOL FOR "THEREFORE"

   $\boxed{g(x) = 2x^2}$

   $\therefore g[\enclose{circle}{f(3)}] = g(4)$ PLUG IN 4 FOR x IN $g(x)$.

   $= 2(4)^2$

   $= 2 \times 16$

   $= \underline{32}$

   (b) $\boxed{f(x) = x + 1}$ PLUG IN $2x^2$ FOR x IN $f(x)$.

   $\therefore f[g(x)] = f(2x^2)$

   $= (2x^2) + 1$

   $= \underline{2x^2 + 1}$

Exercise #11, page 47

1. (a) no; $y = 4$ (constant function)

   (b) yes; $y = x$ ($m = 1$ and $b = 0$)

   (c) yes; $y = \frac{-1}{2}x + \frac{3}{2}$ $\left(m = \frac{-1}{2},\ b = \frac{3}{2}\right)$

   (d) no; $x = -3$ (not a function)

   (e) no; $y = x^2$ (quadratic equation)

   (f) yes; $y = -x + 5$ ($m = -1$, $b = 5$)

2. (a) $\{(x,y)\,|\,y = x - 5\}$; $m = 1$, $b = -5$

   (b) $\{(x,y)\,|\,y = \frac{2}{3}x - \frac{4}{3}\}$; $m = \frac{2}{3}$, $b = \frac{-4}{3}$

   (c) $\{(x,y)\,|\,y = \frac{4}{3}x\}$; $m = \frac{4}{3}$, $b = 0$

   (d) $\{(x,y)\,|\,y = \frac{2}{5}x + \frac{7}{5}\}$; $m = \frac{2}{5}$, $b = \frac{7}{5}$

Exercise #12, page 52

1. (a) $\frac{2-(-1)}{3-(-2)} = \frac{2+1}{3+2} = \frac{3}{5}$ (b) $\frac{(-1)-3}{2-(-1)} = \frac{-1-3}{2+1} = \frac{-4}{3}$

(c) $\frac{3-3}{1-(-3)} = \frac{3-3}{1+3} = \frac{0}{4} = 0$ (d) $\frac{(-3)-3}{1-(-1)} = \frac{-3-3}{1+1} = \frac{-6}{2} = -3$

(e) $\frac{3-0}{2-(-2)} = \frac{3-0}{2+2} = \frac{3}{4}$ (f) $\frac{3-(-2)}{-4-(-4)} = \frac{3+2}{-4+4} = \frac{5}{0}$

(undefined)

2. (a) $y = 3x - 2$ (b) $y = -x + 6$ (c) $y = 2x$

3.

(a) $2x - y = 1$	$y = 2x - 1$	2	-1
(b) $y + 3x = -5$	$y = -3x - 5$	-3	-5
(c) $y = 3 - \frac{1}{2}x$	$y = \frac{-1}{2}x + 3$	$\frac{-1}{2}$	3
(d) $2x + 4y = 6$	$y = \frac{-1}{2}x + \frac{3}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$
(e) $6x - 4y = 7$	$y = \frac{3}{2}x - \frac{7}{4}$	$\frac{3}{2}$	$\frac{-7}{4}$
(f) $-2x - 3y = 6$	$y = \frac{-2}{3}x - 2$	$\frac{-2}{3}$	-2
(g) $5 = 2x - y$	$y = 2x - 5$	2	-5
(h) $x - y = 1$	$y = x - 1$	1	-1
(i) $y + 4x = 0$	$y = -4x$	-4	0

Exercise #13, page 55

1. (a) $m = 2$ and $(x_1, y_1) = (-2, 5)$

$$\begin{aligned} y - \underset{\downarrow}{y_1} &= \underset{\downarrow}{m}(x - \underset{\downarrow}{x_1}) \\ y - 5 &= 2[x - (-2)] \\ y - 5 &= 2(x + 2) \\ y - 5 &= 2x + 4 \\ &\underline{y = 2x + 9} \end{aligned}$$

(b) $m = -1$ and $(x_1, y_1) = (0, 3)$

$$\begin{aligned} y - \underset{\downarrow}{y_1} &= \underset{\downarrow}{m}(x - \underset{\downarrow}{x_1}) \\ y - 3 &= -1(x - 0) \\ y - 3 &= -1(x) \\ &\underline{y = -x + 3} \end{aligned}$$

Exercise #13, continued

(c)

$$\begin{aligned} m &= \frac{\text{difference of ordinates}}{\text{difference of abscissas}} \\ &= \frac{-1 - (-4)}{-2 - (-6)} \\ &= \frac{-1 + 4}{-2 + 6} \\ &= \frac{3}{4} \end{aligned}$$

$m = \frac{3}{4}$ and $(x_1, y_1) = (-6, -4)$ (OR) $m = \frac{3}{4}$ and $(x_1, y_1) = (-2, -1)$

$$\begin{array}{ccccccc} y\ - & y_1 & = & m & (x\ - & x_1 &) \\ & \downarrow & & \downarrow & & \downarrow & \\ y\ - & (-4) & = & \frac{3}{4} & [x\ - & (-6) &] \end{array}$$

$$\begin{aligned} y + 4 &= \frac{3}{4}(x + 6) \\ y + 4 &= \frac{3}{4}x + \frac{9}{2} \\ y &= \frac{3}{4}x + \frac{9}{2} - 4 \\ \underline{y} &\underline{= \frac{3}{4}x + \frac{1}{2}} \end{aligned}$$

$$\begin{array}{ccccccc} y\ - & y_1 & = & m & (x\ - & x_1 &) \\ & \downarrow & & \downarrow & & \downarrow & \\ y\ - & (-1) & = & \frac{3}{4} & [x\ - & (-2) &] \end{array}$$

$$\begin{aligned} y + 1 &= \frac{3}{4}(x + 2) \\ y + 1 &= \frac{3}{4}x + \frac{3}{2} \\ y &= \frac{3}{4}x + \frac{3}{2} - 1 \\ \underline{y} &\underline{= \frac{3}{4}x + \frac{1}{2}} \end{aligned}$$

End of Lesson G

Lessons H-K
and
Answer Key

Mathematics
Upgrading

Math Upgrading

Part B

Distance
Learning

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